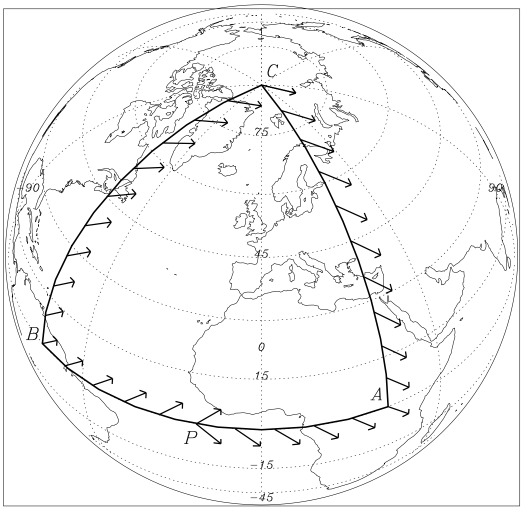
The South-seeking chariot on curved surfaces

Extract

of

D.-E.Liebscher: Einstein's relativity and the geometries of the plane

The curvature of a surface yields a characteristic rotation of directions which are conserved as true as possible on a closed path. This rotation can be seen in fig.8.1 too. We consider the path from P through A, C and B back to P again. If we look at the starting point P south-east, this direction has an inclination to the intended path of $\pi/4$ to the right. It is this angle which we hold fixed till we arrive at A. Here, our path turns left. In oder to conserve our direction, we keep an inclination of $3\pi/4$ to the right, still to south-east. After the next turn at C, our chosen direction has an inclination of $3\pi/4$ to the left. For the next part of the path, this is north-east. After the third turn at B the inclination is $\pi/4$ to the left. We arrive at P and state that the chosen direction has been rotated $\pi/2$ to the left, although we did our best to conserve it. Parallel transport along a closed path produces a rotation. The amount of this rotation is proportional to the curvature and the area enclosed by the path.



At point P we choose an arbitrary direction (here SE). Along the geodesics to point A, from A to C, from C to B and from B back to P we keep the inclination relative to the direction of motion. Only in the points A, C and B we take into account that the path turns about $\pi/2$ to the left and that this angles adds to the inclination of the sight direction to the direction of motion, accordingly. After returning to P, we added to the inclination only $3\pi/2$ to the right. In total, we observe a turn of the tangential plane by $\pi/2$ to the left. This value is identical to the excess of the sum of angles of the triangle ACB. This sum exceed the euclidean value (π) just by these $\pi/2$.

Figure 0.1: A triangle on a sphere with excess and rotation of the tangents

Surprisingly, a machinery exists which realises the geodesic parallel transport of a chosen direction. This is the **South Seeking Chariot** (Zhǐ nán chē) which can be inspected sometimes at exhibitions of ancient Chinese technics (fig. 8.3). A subtracting differential gear guides the central vertical axis so that the direction of the pointer remains fixed even when the chariot under it is turned around on a plane (fig.8.4). The rotation of the indicated direction induced by the curvature of the surface is realized by the pointer too, if the chariot is tracked along the curve $PACBP$. – It has to be noticed that a magnetic needle does *not* accomplish the geodesic parallel transport. On a great circle, the *geodesic* parallel transport which we defined above keeps the angle between the pointer and the direction of geodesic motion constant. The **magnetic needle** defines a different parallel transport. Such a needle keeps the direction to the (magnetic) north pole independent of the particular great circle on which we are moving. The curves of constant inclination against the magnetic needle are loxodromes, not geodesics. We see the difference in stating that the angle of this direction to the direction of the geodesic is changed permanently. – Geodesics are defined to be the shortest connections[1].

[1]Strictly speaking, geodesics are extremal connections. Depending on the local type of the metric properties (Euclidean or pseudo-Euclidean) they are shortest or longest connections. In our example – on the surface of a sphere in an Euclidean space – each path can be made longer by including a small local deviation. Consequently, an extremal path can only be the shortest.

Tales claim that the mythical emperor Huang Di and his army found their way through the nebula to defeat some – of course – rebels by using a chariot pointing always the direction to the enemy. In order to show the feasibility of such a chariot, in the Song dynasty a model chariot has been built. Nothing survived except for a description of the construction and the drawing of a jade model of the pointing figure. The model shown in the photo here stands in front of the Taipeh National Museum. Yinan Chin helped me to get the photo.

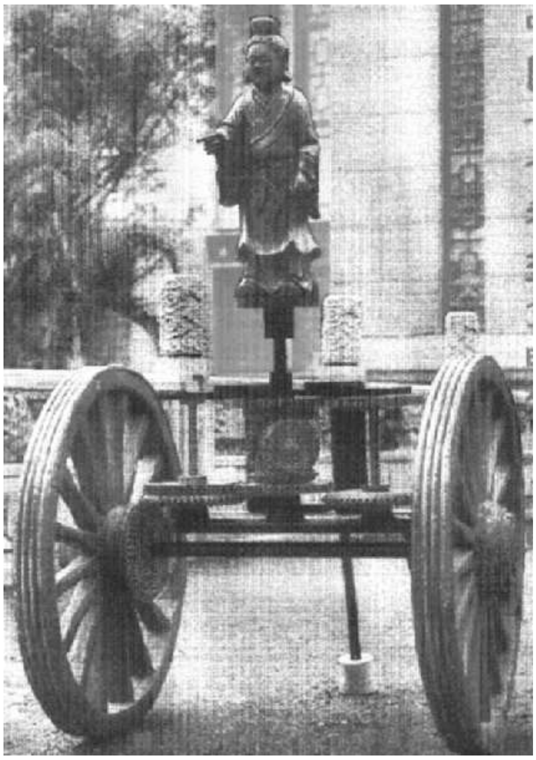


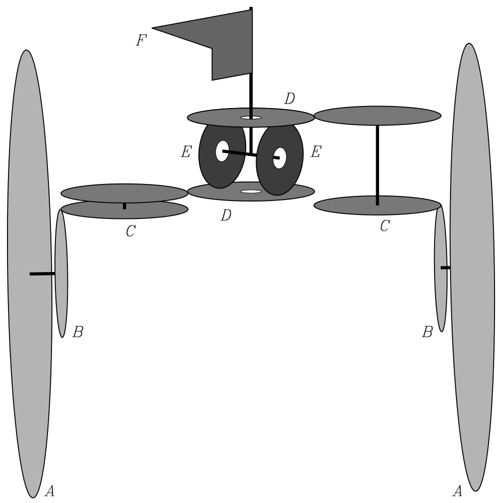
Figure 0.2: A model of a South Seeking Chariot.

For the geodesic parallel transport, they are autoparallel too. Furthermore, they are the only autoparallel curves[2] As we already noted, one can define the parallel transport independent of the metric properties and different from the geodesic parallel transport. In this case it turns out that the curvature is a property of the parallel transport[3]. – In navigating on the surface of the earth, the **gyro compass** renders good service. It keeps the direction to the rotation pole of the earth and is to be compared with the magnetic needle. However, a rotating top is a three-dimensional device. Can we make it formally two-dimensional by forcing its figure axis into the tangent plane? Remarkably, a gyroscope with its axis *constrained* to be tangent to the surface is not appropriate at all to fix a direction. That is because it not only rotates uniformly around its figure axis, but the figure axis rotates around the surface normal too. The latter rotation can be prepared to vanish initially. However, if the direction of the normal is changed along the path on the curved surface the figure axis inevitably sets out to turn by precession. In this case, the result depends f.i. on the velocity of the gyroscope in the surface. In contrast to such an useless *constrained* gyroscope, the gyro compass works thanks to the *free* mobility of its figure axis, to the free support in the earth gravitational field, and to the rotation of the earth. It is an essentially three-dimensional device. – The change of orientation considered here which appears in the parallel transport along closed curves defines the curvature in spaces with more than two dimensions too.

SANTANDER,M. (1992): The chinese south-seeking chariot: A simple mechanical device for visualizing curvature and parallel transport, *Amer.J.Phys.* **60**, 782-787.

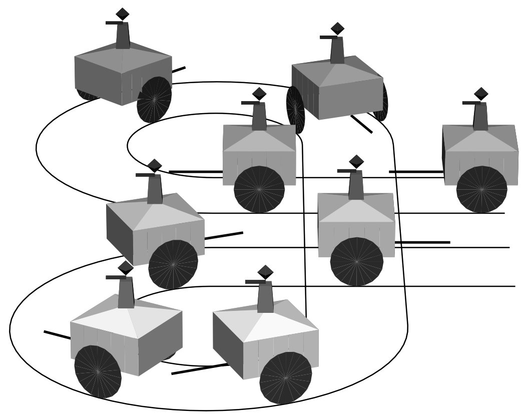
[2]If we always take the direction of the pointer of the South Seeking Chariot, we can be sure that the paths of both wheels have the same length. On the other hand, if the length of the path of the chariot could be shortened by shifting the path sideways, the paths of the two wheels could not be equal and the pointer would indicate this by turning against the chariot. Consequently, the path chosen by the pointer is the shortest possible.

[3]For instance, the parallel transport by the magnetic needle is free of curvature by definition and does not refer to the metric properties of the surface: The length of the path enters nowhere. Now, autoparallel curves differ from geodesics. These subtleties are the background of the unified field theories searched f.i. by A.Einstein, H.Weyl, E.Schrödinger.



A turn of the South Seeking Chariot yields unequal rotation of the two wheels A. Both are tightened to the wheels B which transfer the rotation to the wheels C which obtain a different sign of rotation if the wheels B have the same. This rotation is transferred again to the wheels D which form together with the cursor wheels E the proper differential gear as it is used f.i. in cars. The rotation of the axis of the flag F is half the sum of the rotations of the wheels D. Hence it is half the *difference* of the rotation of the wheels A. It is a question of gear that the rotation of the flag compensates for the turning of the car. Then, on a plane the pointer F indicates always the same direction. If the chariot moves on a curved surface, the pointer preserves the direction locally as well as we did with the line of sight in figure 8.1.

Figure 0.3: The mechanics of a South Seeking Chariot.



With the South Seeking Chariot, the parallel transport of a direction is realized. Independent on how the chariot is drawn along its path, the gears inside provide for the keeping constant of the direction in which the figure on the top is pointing.

Figure 0.4: The motion of a South Seeking Chariot.

Needham,J. (1977): Physics and physical technology II. Mechanical engineering, *Science and civilization in China* **4**, Cambridge UP.

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