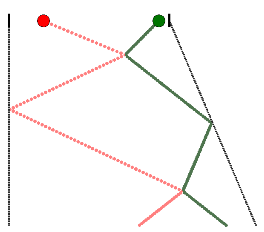
Synthetic geometry and Relativity theory

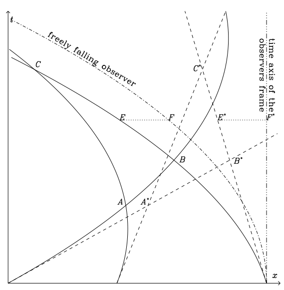
D.-E.Liebscher (Potsdam)
http://www.aip.de/~lie/

It is the intention of this lecture to recall the close connection between elementary geometry and elementary mechanics. This connection is an interdisciplinary tool to visualize aspects of both subjects that remain abstract (i.e. not mapped onto everyday experience) otherwise. The key to this connection is projective geometry. Geometry is experienced in mechanics (and physics in general), and mechanics defines the geometry of space *and* time. It is relativity theory that is of particular public interest. Its curious statements about space and time encourage many people to think about and to map them onto the experience they have. In trying to pin down the arguments you have to draw, i.e., you have to use the geometry of the space and time. Most of us heard about the twin paradox, and most of us know that it represents only the unconventional triangle inequality of pseudoeuclidean space. The statement that relativity theory is the geometry of space-time is commonplace, but synthetic treatments of this geometry are rare (my knowledge about the mathematical literature is incomplete, so I cite only [2, 3, 6, 8, 15, 16, 18, 17, 19, 20]. However, I am sure that in the literature for physics, in particular relativity, there is no textbook that really shows Minkowski geometry in its synthetic form. Most of the books about relativity sketch the problems a bit helpless in space, not in space-time. Apparently, the development of analytic and algebraic methods that required all the attention of the physics community all the twentieth century covered the synthetic aspect with the attribute of being old and slow and of no use at all, a Glass Bead Game.

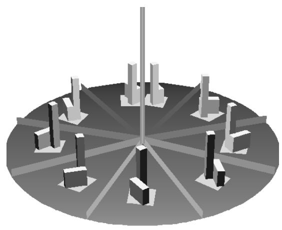
I intend to remind you, that
1. space-time diagrams are in fact commonplace
2. mechanics require space-time constructions, not simply diagrams
3. space-time constructions allow to explicitely show the relation between mechanics and geometry
4. space-time constructions explain geometric arguments simpler than the euclidean plane
5. space-time constructions show the necessity and use of abstract geometric formulation on the level of school geometry.
6. the common descendance of euclidean and pseudoeuclidean geometry from the projective one leads to deeper questions of physics.
7. plane geometry can feature the universe.



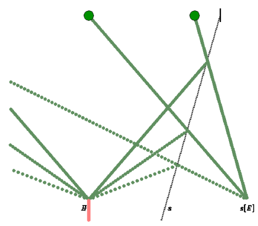
Movie 1: A strip registration



Movie 2: Parabolas of throw are straight lines



Movie 3: The toilet mirror



Movie 4: An explosion

Space-time diagrams are commonplace as paper registrations of time-dependent quantities in medicine, seismology, meteorology. The place of ink needle is recorded, that mostly translates fluctuations or oscillations about some mean value. The evaluation mostly hides the geometric aspect in the algorithmic apparatus. However, these registrations yield the opportunity to envisage the registration of unidimensional motion, and now we are inmidst of elementary mechanics (Movie 1).

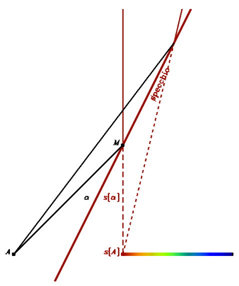
Uniform motion leaves a straight line. We may understand, that being straight does not depend on the factual uniformity of the registration: A line is straight if it is element of a congruence of lines where two lines intersect at most once, and two points determine a unique line. These properties are not hampered when the registration itself is not uniform. We may understand, that the uniformity of registration is controlled by the congruence. Its coordinates must describe the congruence with linear relations. Newtons first law states, that the world-lines of free motion form a set of straight lines in space-time, that defines in its turn linear reference frames, the inertial frames. An unusual, but important example are the world-lines of the vertical throw (Movie 2). Our euclidean habit shows us parabolas, but these parabolas constitute (through their intersection properties) a set of straight lines. We get the conventional linear picture through referring the spatial distance to one of the parabolas: This is the inertial reference to a freely falling observer.

Euclidean geometry has no place on the registration strip. No euclidean circle has any counterpart in the registration of physical motion. What curve on the registration strip plays the role of the circle on the euclidean plane? We find (in the case of force-free motions) only straight lines between collisions or reflections.

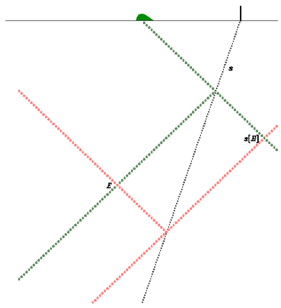
Reflection is the keyword. We learn at school with astonishment, that translations and rotations can be generated through reflections only. The depth of this statement cannot be understood at school, where we are happy to construct reflections with the help of the circle, and where we have no access to the reflection without the notion of rotation. The exception, of course, is the toilet mirror (Movie 3). In space-time, however, we primarily have the reflection, and can infer the circle as the locus of equal distances to some centre.

We start with the ordinary mechanical reflection that inverts the relative velocity without changing its modulus (Movie 4), used by Amélie Mauresmo, for instance. The reflected image of an event is a simultaneous event at equal but opposite spatial distance from the mirror (Movie 5). The circle about an event on the world-line of the mirror is the connecting line: how strange! The distance between two events is the difference in *time*, not in space: how strange! The angle between two straight world-lines is the relative velocity: Velocities are composed additively: this is well-known terrain at last. This is the Galilei geometry [8].

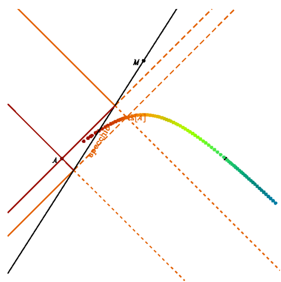
However, additive composition of velocities is only an *approximation* to the real world around us: the velocity of light is *not* changed in composition with other velocities. We have to leave the mechanics argument for constructing reflection and pass to the simpler costruction with waves (Movie 6). This kind of reflection replaces the euclidean rectangular involution on the line at infinity that has no real fixed points with a rectangular involution that has two real fixed points: the carrier of the congruences of world lines of light propagation. Reflections are now linear constructions and do *not* require the use of any second-order curve. We can see now that the reduction of translations and rotations to reflections makes things not only simpler, but *accessible*.



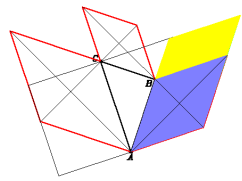
Movie 5: The Galilei circle



Movie 6: Light quadrilaterals



Movie 7: The Minkowski circle



Movie 8: The theorem of Pythagoras

The euclidean circle is replaced with a curve that our euclidean interpretation is a hyperbola (Movie 7). The theorem of Pythagoras yields the famous indefinite signature of the metric (Movie 8). Many of the euclidean strategies of proof (in fact, the strategies that can be reduced to the projective-metric connection) work in a completely analogous way (Movie 9). A curious fact is that distances on lightlike lines vanish: Photons do not age (Movie 10). The composition of relative velocities is the catenation rule for cross ratios (Figure 11).

Cross ratios are the next keyword. Cross ratios as measure – this reminds us that the general case of induction of metric relations in the plane is the definition of an absolute conic. The distance between two points is given through the cross ratio of the pair with the two intersection point of the connecting line with the absolute conic. Angles are defined through the cross ratio with the two tangent lines to the absolute conic. We so entered the realm of Cayley-Klein geometries, and of the Klein model of non-euclidean geometry in particular. It is the part of the plane outside the conic, where all basic constructive elements are real. Timelike lines intersect the conic in two real points, and from the intersection of two timelike lines, two real tangents to the conic exist. The cross ratios can be found without referring to non-real points and lines (Movie 12). Angles between lines that intersect on the conic vanish (these are parallels in the conventional sense) just as distances on lines tangent to the conic vanish: these are the lightlike lines in Minkowski geometry. The part of the plane outside the absolute conic (beyond the infinity of the non-euclidean geometry) is the two-dimensional model of the deSitter universe locally (Figures 13 and 14).

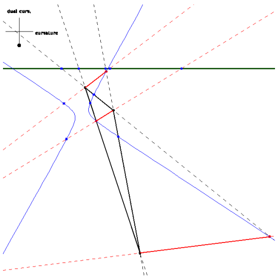
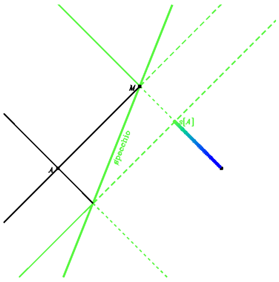


Figure 9: The Feuerbach circle in Minkowski g.



Movie 10: Zero distances on lightlike lines

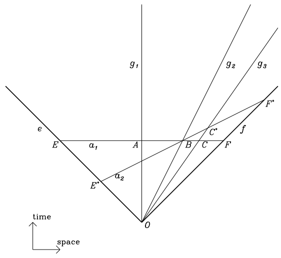
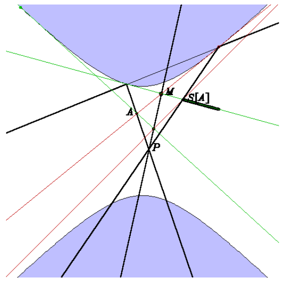


Figure 11: Cross ratios and velocities



Movie 12: Zero angles for praralel lines

A typical problem to show the use of geometric constructions in space-time is the Fresnel paradox. Fresnel discovered that wave-fronts (in the frame of absolute simultanieity) do not show aberration, while particle motion does. The construction shows without any arithmetics that the solution of the paradox requires relativity of simultaneity, i.e. Minkowski geometry for space-time (Figures 15 and 16).

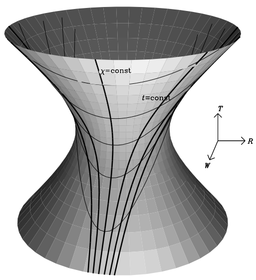


Figure 13: Motion on a pseudosphere

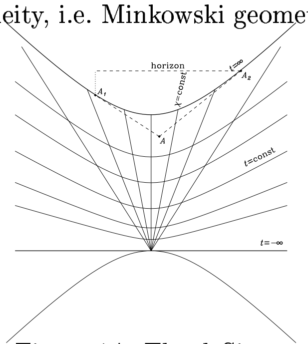


Figure 14 : The deSitter universe

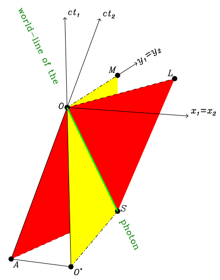


Figure 15: The Fresnel paradox I

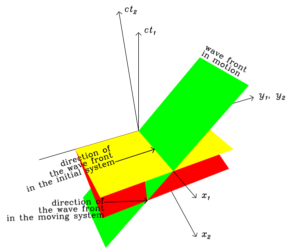


Figure 16: The Fresnel paradox II

Metric properties are induced by the definition of an absolute conic. Is this conic (generally quadric in a fourdimensional space-time) an object of physics? Is the observed metricity to be reduced to an object that has to be explained by dynamics? The point of view of projective geometry leads to a far deeper and unsolved question about the physical origin and reason for the metrisation of space-time. The measure of physics is given by action integrals. Action principles rule mechanics and field theory as well. Action principles rule the state and the motion of our rulers and our clocks. The metric properties of space-time should descend from the action principles. The existence of simple metric properties should depend on the ideal state of the universe [5, 12, 21]. We do not proceed like this for the moment. We always start by assuming the existence of a space-time metric with a structure independent of the physical state of the universe. General relativity provides only the changes of this metric from point to point, not its existence, and not its signature. We form action principles only with these geometric premises: no surprise that dynamics fits with this supposition. However, the question where the metric property of space-time comes from cannot be asked. Nontheless, I hope one can understand eventually whether one can establish a connection between the measure of dynamics with the measure of space and time without supposing the latter beforehand. But this is another topic, and another lecture.

For this time, I restrict myself to the projective-metric connection, and I thank you four your attention.

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