GRAVITATIONAL SINGULARITIES VIA ACCELERATION: THE CASE OF THE SCHWARZSCHILD SOLUTION AND BACH'S GAMMA METRIC

SALVATORE ANTOCI, DIERCK-EKKEHARD LIEBSCHER, AND LUIGI MIHICH

ABSTRACT. The so called gamma metric corresponds to a two-parameter family of axially symmetric, static solutions of Einstein's equations found by Bach. It contains the Schwarzschild solution for a particular value of one of the parameters, that rules a deviation from spherical symmetry. It is shown that there is invariantly definable singular behaviour beyond the one displayed by the Kretschmann scalar when a unique, hypersurface orthogonal, timelike Killing vector exists. In this case, a particle can be defined to be at rest when its world-line is a corresponding Killing orbit. The norm of the acceleration on such an orbit proves to be singular not only for metrics that deviate from Schwarzschild's metric, but also on approaching the horizon of Schwarzschild metric itself, in contrast to the discontinuous behaviour of the curvature scalar.

1. INTRODUCTION

In the early days of general relativity there was little doubt that in the new theory the Christoffel symbols had the role of "components" of the gravitational field [1, 2]. Einstein was very explicit on this point: at page 802 of his paper [2] on the foundations of general relativity, he says:

> Verschwinden die $\Gamma^{\tau}_{\mu\nu}$, so bewegt sich der Punkt geradlinig und gleichförmig; diese Größen bedingen also die Abweichung der Bewegung von der Gleichförmigkeit. Sie sind die Komponenten des Gravitationsfeldes.[1]

This identification can be deemed satisfactory if only gravitation and inertia are considered: it accounts for the shifty role that the gravito-inertial field plays in the geodesic equations of motion for a pole test particle whose

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[1] If the $\Gamma^{\tau}_{\mu\nu}$ vanish, the point moves uniformly on a straight line; therefore these quantities cause the deviation of the motion from uniformity. They are the components of the gravitational field. (It should be added that from the projective point of view a straight line is defined by being element of any set of lines that intersect at most once and that are determined by two distinct events. Hence any set of coordinate lines will do. The ambiguity of this set is usually reduced by choosing a set of geodesic lines, at the expense of the restriction of the intersection axiom to only local validity.)

four-velocity is u^i:

$$\frac{\mathrm{d}u^i}{\mathrm{d}s} + \Gamma^i_{kl} u^k u^l = 0. \tag{1.1}$$

The coefficients of the affine connection are not tensorial. Hence any identification with gravitation or inertia depends on the choice of the coordinate system. For particular world lines defined by the solution itself, this drawback may be partly reduced, as we shall see later. Equation (1.1) can be however read as a particular case of the law

$$\frac{\mathrm{d}u^i}{\mathrm{d}s} + \Gamma^i_{kl} u^k u^l = a^i. \tag{1.2}$$

This is the general relativistic extension of Newton's second law; it can be derived from the conservation identities of the theory [2], and claims that the force per unit mass a^i exerted by non-gravitational fields on the pole test particle is balanced by the gravito-inertial pull (per unit mass) expressed by the left-hand side of (1.1). Therefore, in order to provide a definition of the gravito-inertial force, it is not enough to know the metric g_{ik}, hence the Christoffel symbols: as stressed by Whittaker [4], "in general relativity the gravitational force, as measured by any observer, depends not only on the observer's position but also on his velocity and acceleration", being in fact given by the four-vector a^i of equation (1.2). The relativist of the present day, convinced of the exclusive role that invariant entities play in general relativity, may feel some relief in noticing that the gravitational pull, defined by Einstein in terms of noninvariant quantities, according to Whittaker is accounted for by a tensorial entity, associated with the world-line followed by the pole test particle. Furthermore, in the cases when this world-line for some reason, like some symmetry of the metric, has not a contingent character, but can be uniquely chosen through an invariant definition, the norm

$$\alpha \equiv (-a_i a^i)^{1/2} \tag{1.3}$$

of the four-acceleration will provide a fully invariant and maybe physically relevant measure of the otherwise elusive gravito-inertial force.

However, the evident merits of the definition of the gravito-inertial pull adopted by Whittaker were not considered by Synge [5] to be sufficient for overcoming the alleged drawback, that the gravito-inertial force defined in this way happens to vanish when the pole test particle undergoes geodesic motion. Moreover, a change in attitude appeared to him necessary: in Einstein's theory only relative kinematic measurements involving nearby particles are permitted in general, hence one must renounce the unattainable goal of determining absolutely the force acted by the gravitational field on any particle, and must be content with a differential law that only allows for the comparison of the gravitational pull acting at adjacent events. Let

[2] For the case of an electrically charged test particle see *e.g.* [3].

us forget about nongravitational forces and their balance with the gravito-inertial ones: we consider two pole test particles executing geodesic motion, and imagine that their world-lines L, M be very close to each other. If η^i is the infinitesimal displacement vector drawn perpendicular to L from a point A on L to a point B on M, the acceleration of B relative to A is defined as the vector

$$f^i = \frac{\mathrm{D}^2\eta^i}{\mathrm{d}s^2}, \tag{1.4}$$

where $\mathrm{D}/\mathrm{d}s$ indicates absolute differentiation and $\mathrm{d}s$ is the infinitesimal arc length of the geodesic L measured at A. But Synge himself [6],[7] has proved that

$$\frac{\mathrm{D}^2\eta^i}{\mathrm{d}s^2} + R^i{}_{jkl}u^j\eta^k u^l = 0, \tag{1.5}$$

where $R^i{}_{jkl}$ is the Riemann tensor and u^i is the four-velocity of the particle at A. By appealing to Newton one then postulates that the excess of the gravitational force at the event B over the gravitational force at the event A is naturally defined (for unit test masses) to be the acceleration of B relative to A. Therefore one finds [5] that the excess of the gravitational pull is given by

$$f^i = -R^i{}_{jkl}u^j\eta^k u^l. \tag{1.6}$$

Whittaker's definition of the gravitational pull through the four-acceleration of a single pole test particle in arbitrary motion is superseded by a relative definition, in terms of the relative four-acceleration of two adjacent pole test particles both executing a geodesic motion. Whittaker's definition looks like the prolongation, within general relativity, of a line of thought that can be traced back to Newton and d'Alembert, and gives the acceleration four-vector a fundamental physical role. Synge's definition of the relative pull is instead one of the first attempts at building the physical interpretation of the geometric structure of general relativity by availing as few as possible of concepts inherited from the physics of the past, and happens to give a central role to the Riemann tensor itself. One should not think of this change of attitude as an abrupt occurrence in the development of general relativity: despite an eloquent counterexample [8],[9] exhibited by Levi-Civita, the idea that a "true" gravitational field must necessarily entail a nonvanishing Riemann tensor is already present in Eddington's book of 1924. In that book Eddington [10] proposes to tell apart the fake gravitational waves, *i.e.* mere undulations of the coordinate system that can propagate with the "speed of thought", from the alleged true ones by looking at the behaviour of the Kretschmann scalar $R^i{}_{jkl}R_i{}^{jkl}$. Whittaker's definition of the gravitational pull is however still alive and well in a paper written by Rindler [11] in 1960. Nevertheless, a quiet revolution has indeed occurred in general relativity if, in the section entitled "Gravitational field" of his essay [12] in honour of the geometer Hlavatý, Synge could eventually assert that the wise plan is

to forget about Newton's arrow and say "gravitational field = curvature of space-time".

One cannot help noticing that this assertion, of course impressive and general in character, is much vaguer than the precise and somewhat technically unwieldy definition (1.6) of the relative gravitational pull given by Synge in 1937. Nevertheless this assertion, despite Levi-Civita's counterexample [8], has entered the mind habit of the relativist: whenever asked about the properties of the gravitational field associated with a given metric, in particular about its possible singularities, one immediately thinks to the Riemann tensor or, better, to the invariants that can be constructed from it.

2. The case of the gamma metric

Of course, the identification of a geometric entity of Einstein's general relativity with a physical one has not to be decided *a priori* on the basis of some preconception about the best way for building theoretical physics; a study of how the proposed identification works in clear-cut examples provided by the theory is a necessary requisite for settling such issues. In the present paper we shall compare the insight in the behaviour of the singularities of the gravitational field that can be obtained through the Whittaker definition and through the Riemann tensor approach in the particular case of the so called gamma metric.

The latter is one of the axially symmetric, static solutions [13] calculated in 1922 by Bach, who availed of the general method [14],[15] found by Weyl and by Levi-Civita. Despite the nonlinear structure of Einstein's equations, Weyl succeeded in reducing the axially symmetric, static problem to quadratures through the introduction of his "canonical cylindrical coordinates". Let $x^0 = t$ be the time coordinate, while $x^1 = z$, $x^2 = r$ are the coordinates in a meridian half-plane, and $x^3 = \varphi$ is the azimuth of such a half-plane; the adoption of Weyl's canonical coordinates allows writing the line element of a static, axially symmetric field *in vacuo* as:

$$\mathrm{d}s^2 = e^{2\psi}dt^2 - \mathrm{d}\sigma^2, \tag{2.1}$$
$$e^{2\psi}\mathrm{d}\sigma^2 = r^2\mathrm{d}\varphi^2 + e^{2\gamma}(\mathrm{d}r^2 + \mathrm{d}z^2);$$

the two functions ψ and γ depend only on z and r. Remarkably enough, ψ must fulfil the "Newtonian potential" equation

$$\Delta\psi = \frac{1}{r}\left\{\frac{\partial(r\psi_z)}{\partial z} + \frac{\partial(r\psi_r)}{\partial r}\right\} = 0, \tag{2.2}$$

where ψ_z, ψ_r are the derivatives with respect to z and to r respectively, while γ is obtained by solving the system

$$\gamma_z = 2r\psi_z\psi_r,\ \gamma_r = r(\psi_r^2 - \psi_z^2); \tag{2.3}$$

due to the potential equation (2.2)

$$\mathrm{d}\gamma = 2r\psi_z\psi_r\mathrm{d}z + r(\psi_r^2 - \psi_z^2)\mathrm{d}r \tag{2.4}$$

happens to be an exact differential.

The particular Bach's metric we are interested in is defined by choosing for ψ the potential that, in Weyl's "Bildraum", is produced by a thin massive rod of constant linear density $\sigma = k/2$ lying on the symmetry axis, say, between $z = z_2 = -l$ and $z = z_1 = l$. One finds:

$$\psi = \frac{k}{2} \ln \frac{r_1 + r_2 - 2l}{r_1 + r_2 + 2l}, \tag{2.5}$$

where

$$r_i = [r^2 + (z - z_i)^2]^{1/2}, \ i = 1, 2. \tag{2.6}$$

By integrating equations (2.3) and by adjusting an integration constant so that γ vanish at the spatial infinity one obtains:

$$\gamma = \frac{k^2}{2} \ln \frac{(r_1 + r_2)^2 - 4l^2}{4r_1 r_2}. \tag{2.7}$$

The resulting metric is asymptotically flat at spatial infinity and its components are everywhere regular, with the exception of the segment of the symmetry axis for which $z_2 \leq z \leq z_1$, for any choice of the parameters l and k, assumed here to be positive.

It may be convenient [16] to express the line element in spheroidal coordinates by performing, in the meridian half-plane, the coordinate transformation [17]:

$$\varrho = \frac{1}{2}(r_1 + r_2 + 2l), \ \cos\vartheta = \frac{r_2 - r_1}{2l}. \tag{2.8}$$

Then the interval takes the form

$$ds^2 = \left(1 - \frac{2l}{\varrho}\right)^k \mathrm{d}t^2 - \left(1 - \frac{2l}{\varrho}\right)^{-k} \cdot \left[\left(\frac{\Delta}{\Sigma}\right)^{k^2-1} \mathrm{d}\varrho^2 + \frac{\Delta^{k^2}}{\Sigma^{k^2-1}} \mathrm{d}\vartheta^2 + \Delta \sin^2\vartheta \mathrm{d}\varphi^2\right], \tag{2.9}$$

where

$$\Delta = \varrho^2 - 2l\varrho, \ \Sigma = \varrho^2 - 2l\varrho + l^2 \sin^2\vartheta. \tag{2.10}$$

We notice that when $k = 1$ the metric reduces to Schwarzschild's spherically symmetric one. It does so in the strict sense, *i.e.* it is in one-to-one correspondence with Schwarzschild's original solution [18], not with the "Schwarzschild metric" of all the manuals and research papers, that was actually found by Hilbert [19]. The latter metric would be retrieved from (2.9) with $k = 1$ if the radial coordinate ϱ were allowed the range $0 < \varrho < \infty$ while, due to (2.8), the allowed values of ϱ are presently in the range $2l < \varrho < \infty$.

Let us now investigate the singularities of the gravitational field of the gamma metric by adopting the point of view according to which "gravitational field = curvature of space-time". We shall explore the singular behaviour of the Riemann tensor in an invariant way through the Kretschmann

scalar $K = R^{i}{}_{jkl} R_{i}{}^{jkl}$. The calculation and the study of this scalar has been done long ago [17] by Cooperstock and Junevicus, and was recently repeated by Virbhadra. We quote here his result [20], expressed with spheroidal coordinates:

$$K = \frac{16 l^2 k^2 N}{\varrho^{2(k^2+k+1)} (\varrho - 2l)^{2(k^2-k+1)} \Sigma^{3-2k^2}}, \tag{2.11}$$

with

$$\begin{aligned} N = {} & l^2 \sin^2\theta \cdot \\ & \left[3lk(k^2+1)(l-\varrho) + k^2(4l^2 - 6l\varrho + 3\varrho^2) + l^2(k^4+1)\right] \\ & + 3\varrho[(k+1)l - \varrho]^2(\varrho - 2l). \end{aligned} \tag{2.12}$$

We do not need a minute analysis of the function $K(\rho, \vartheta, k)$ to decide that, for monitoring the singularities of the gravitational field in the gamma metric, the Kretschmann scalar is a discontinuous parameter. It is sufficient to examine the behaviour of K in the neighbourhood of $k = 1$, *i.e.* for small axially symmetric deviations from spherical symmetry. By studying the zeroes of both the numerator and the denominator of (2.11) one is then confronted with the following situation. For all the values of ϑ, the denominator vanishes when $\rho \to 2l$, while the values of ϱ at which the zeroes of the numerator occur depend on both k and ϑ. As a consequence in the neighbourhood of $k = 1$ the Kretschmann scalar always diverges [17] for $\rho \to 2l$, provided that $k \neq 1$. When $k = 1$, *i.e.* when the metric reduces to Schwarzschild's one, both the numerator and the denominator tend to zero as $\rho \to 2l$ for all ϑ, and fatefully they do so in such a way that the limit value of the Kretschmann scalar happens to be finite at the "Schwarzschild radius".

It is evident that, since the slightest deviation from spherical symmetry restores the divergence of the Kretschmann scalar for $\rho \to 2l$, the Kretschmann scalar is a discontinuous function on the set of Bach metrics defined by equations (2.5) and (2.7). This behaviour complies, in the particular case of the gamma metric, with a well known theorem on the event horizons in static vacuum spacetimes [21].

Let us adopt now the viewpoint on the gravitational pull considered by Whittaker [4]. Since the gamma metric is static, Whittaker's approach allows in this case for a fully invariant and physically transparent treatment. In fact, not only the norm (1.3) of the four-acceleration is a scalar, but also an invariantly prescribed and physically privileged choice of the world lines of the test particles is possible. This choice is not arbitrarily imposed on the manifold, but is dictated by the manifold itself in a purely local way. For choosing these world lines one can in fact avail of the symmetries of the gamma metric, and adopt the unique timelike Killing congruence that not only enjoys the Killing property, but is also hypersurface orthogonal. Either in canonical or in spheroidal coordinates this invariantly distinguished congruence is identified by the constancy of the spatial coordinates. In

this case Whittaker's proposal both fulfils the general requirement, that in general relativity the relevant physical properties must be definable in invariant form, and possesses a straightforward meaning, in keeping with time-honoured ideas of pre-relativistic physics [22]. The norm α of the four-acceleration measures in fact the strength of the gravitational pull exerted on a pole test particle of unit mass kept at rest in the given field.

For the static, axially symmetric interval of equation (2.1) the acceleration four-vector a^i of a test particle kept at rest (z, r, φ constant) has the nonvanishing components

$$a^1 = \exp(2\psi - 2\gamma)\frac{\partial \psi}{\partial z},\ a^2 = \exp(2\psi - 2\gamma)\frac{\partial \psi}{\partial r}, \tag{2.13}$$

hence its squared norm, in the particular case of the gamma metric, reads

$$\alpha^2 = \frac{16k^2l^2}{(4r_1r_2)^{1-k^2}(r_1+r_2-2l)^{1-k+k^2}(r_1+r_2+2l)^{1+k+k^2}}, \tag{2.14}$$

when referred to Weyl's canonical coordinates, and

$$\alpha^2 = \frac{16k^2l^2}{[4(\varrho-l)^2-4l^2\cos^2\vartheta]^{1-k^2}(2\varrho-4l)^{1-k+k^2}(2\varrho)^{1+k+k^2}}, \tag{2.15}$$

when expressed as a function of the spheroidal coordinates of equation (2.8). For any value of $k < 2$ and for all ϑ the norm α happens to grow without limit as $\varrho \to 2l$. At variance with what occurs with the Kretschmann scalar, no discontinuous behaviour of α is noticed when crossing the value $k = 1$, for which the gamma metric acquires spherical symmetry.

3. Conclusion

In the case of the gamma metric two definitions of the gravitational field were considered with respect to their use as indicators of physically meaningful singularities. The behaviour of the Kretschmann scalar is continuous only in carefully chosen sets of solutions; in the set of gamma metrics, it is not. On the contrary, the norm α of the four-acceleration of a test particle kept at rest, that according to Whittaker should provide the strength of the gravitational field measured on a unit mass at rest in this static metric, appropriately vanishes for $\varrho \to \infty$, is everywhere finite for $2l < \varrho < \infty$, and uniformly diverges in the limit $\varrho \to 2l$ for all the values of the parameter k in a suitable neighbourhood of $k = 1$.

In the existing literature the finite value of the Kretschmann scalar at the "Schwarzschild radius" is adduced as an argument for the viability of the program of analytic extension [23],[24] of the Schwarzschild solution. The existence of this extension is an isolated property, and it does not change the singular behaviour of the invariantly defined Whittaker's force.

The uniformly divergent behaviour of the norm α when $\varrho \to 2l$ confirms the wisdom of Schwarzschild's deliberate choice [18] to remove to the "Nullpunkt", *i.e.* to the border of the manifold considered by him, the singular two-surface that brings his name.

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Dipartimento di Fisica "A. Volta" and INFM, Pavia, Italy
E-mail address: Antoci@fisicavolta.unipv.it

Astrophysikalisches Institut Potsdam, Potsdam, Germany
E-mail address: deliebscher@aip.de

Dipartimento di Fisica "A. Volta" and INFM, Pavia, Italy
E-mail address: Mihich@fisicavolta.unipv.it