Purely affine theories with matter

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We try to find a way to obtain *the very existence* of a space–time metric from an action principle that does not refer to it a priori. Although there are reasons to believe that only a non–local theory can viably achieve this goal, we investigate here local theories that start with Schrödinger's purely affine theory [36], where he gave reasons to set the metric proportional to the Ricci curvature aposteriori. When we leave the context of unified field theory, and we couple the non–gravitational matter using some weak equivalence principle, we can show that the propagation of shock waves does not define a lightcone when the purely affine theory is local and avoids the explicit use of the Ricci tensor in realizing the weak equivalence principle. When the Ricci tensor is substituted for the metric, the equations seem to have only a very limited set of solutions. This backs the conviction that viable purely affine theories have to be non–local.

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I. INTRODUCTION

Purely affine theories became a topic of Relativity through the search for a theory unifying gravitation and electromagnetism. The central question of a purely affine theory is the generation of a metric in the course of the evaluation of the field equations. Just this point is essential for the discussion of a relativistic form of Mach's principle, that still poses open questions in GRT. When we consider the possibility to construct a background theory for the existence of the metric itself, we have to revise the features that local purely affine theories present. The a posteriori generation of the space–time metric is a central issue for a relativistic implementation of a Mach-type symmetry breakdown to locally Lorentz invariant theories [22–24]. The implementations of the Mach principle into a relativistic theory of gravity found different aspects and different directions to explore [3, 39]. Considering a Mach type symmetry breakdown to locally Lorentz invariant theories, the important aspect is that the light-cone is the structure that should be generated through that break–down. This implies that the metric structure itself should not enter the gravitation theory a priori. The metric structure, together with the existence of a light–cone, should be the outcome of the theory. In this context, the distribution of matter in the surrounding universe represents the classical vacuum for the local neighbourhood that breaks the at least affine invariance of vector spaces to the Lorentz invariance; for a local breakdown, see Ref. [25]. This is the reason why we do not use the in other respect successful path to extend the metric theory to a metric-affine theory [14], but return again to purely affine theories.

At this point, the metric is only expected to be a second-order tensor that appears in the simplest equations of motion, like the motion of pole particles or the propagation of shock waves of any field. These equations should be compared with the corresponding equations of GRT in order to identify the light-cone structure, or the projective structure used by Ehlers,Pirani and Schild [11]. Here we shall consider shock waves because the appearance of the light-cone structure is our central point. These shockwaves can be matter shocks as well as pure gravitational shocks. In any case, we need only the simplest approximations. Symmetry properties of the tensor that is to be identified as metric are a second-order problem and will not be discussed here.

The first observation is that we cannot avoid the use of a connection $\Gamma^a{}_{bc}$ even when the use of an ordinary metric is avoided: The pure definition of a covariant derivative requires its existence. Then, we have two options. First, we can formulate the problem to find a metric as some solution to the Weyl-Cartan problem: To find a second–order covariant tensor g_{ik} that is covariantly constant with respect to the transport $\Gamma^a{}_{bc}$ [6, 7]. The second option is to find independently this tensor from field equations, and consider its relation to the $\Gamma^a{}_{bc}$ afterwards. We choose this latter option here, because we intend to study the possibility of a dynamical definition of the metric. Its definition through the Weyl–Cartan space problem is a priori to the construction of the coupling to matter and not a posteriori. We are interested in a scheme that constructs the action without use of the Riemannian metric so that the latter can arise from dynamics, i.e. a posteriori.

We hence start from a theory that defines gravitation by a connection field. The question is how to couple external fields and how to get the notion of a metric a posteriori, i.e. to find the equivalent for the metric tensor. We expect that this a posteriori metric tensor is defined only to some approximation, or as a result of some symmetry breaking process, and that its precise defineability requires particular configurations of the gravitational field.

Let us construct the simplest action integral for some fields Φ^A, where A stands for any field components without referring to their quality as scalar, vector, or tensor of any rank. The question of spinors in purely affine theory requires particular attention, see for instance Ref. [8, 25]. First, we look for a second–order field theory, i.e. for an action bilinear in the derivatives, $\Phi^A{}_{,k}$. However, covariance requires the use of covariant derivatives, $\Phi^A{}_{;k}$, which are defined through some linear connection $\Gamma^m{}_{nk}$. The correction to the ordinary derivative for obtaining the covariant one is linear in the coefficients of the connection and linear in the field,

$$\Phi^A{}_{;k} = \Phi^A{}_{,k} + C^A{}_B{}^n{}_m\ \Gamma^m{}_{nk}\Phi^B. \tag{1}$$

where $C^A{}_B{}^n{}_m$ are some coefficients to be determined and depend on the nature of the matter fields. For instance, when A stands for indexing the components of a contravariant vector, then $C^A{}_B{}^n{}_m = \delta^A_m \delta^n_B$. In metric-affine theories, the interpretation of the torsion and non-metricity part of the $\Gamma^a{}_{bc}$ is a famous problem [4, 9, 14, 41].

Second, the integrand must be a scalar density, so the indices of derivation have to be compensated by some appropriate construction that provides upper indices. In the General Relativity theory (GRT) this is done through the metric tensor, more precisely, through its contravariant inverse, and combinations of it. Here, we have two options that are characterized through the use of the Ricci tensor. We shall consider them below.

Third, we need an invariant volume element [12]. When there is no metric, and hence no determinant of the metric tensor, the simplest choice is the determinant of the Ricci tensor, as A.S. Eddington pointed out in the early 1920's. This is Schrödinger's choice too [10, 32–36]. With the simple action,

$$S_1 = \int \sqrt{-\det\ R_{ab}}\ \ d^4x\ , \tag{2}$$

where R_{ab} denotes the Ricci tensor. This was used already by Eddington [10], but with the restriction to a symmetric connection. Schrödinger obtained a theory for the general affine connection that suggested to equate the Ricci tensor with the metric: The Ricci tensor obeys a field equation that tells that it is covariantly constant with respect to the star affinity up to the torsion of the latter. Therefore, Schrödinger postulated that

$$g_{ik} = \frac{1}{\lambda} R_{ik}\ . \tag{3}$$

For an unified field theory that does not explicitly contain matter this interpretation might be satisfying, but for a theory with explicit matter terms it is not. Indeed, Schrödinger's original intention was to get a unified field theory with no external matter at all, and the problem was to find equivalents for the conventional matter first. In the present work we assume the gravity sector given by that of Schrödinger's and see how and which ordinary matter can be coupled to such a gravitation.

We show that it is not sufficient to purely determine the Ricci tensor to act as metric. The metric that is inferred by observation is that of the motion of matter [9]. This is also the lesson in particular of all theories with more than one metric tensor [24]. Explicit matter defines an effective metric by its motion, either by the motion of test particles or by the motion of shock waves. We have to define test particles of the matter fields that allow to construct an effective metric through the Ehlers–Pirani–Schild procedure [11], or we have to consider the propagation of shock fronts that only for gravitation pose a particular problem [30, 38]. We underline that it is the non-gravitational fields and particles that are used to determine the gravitational field, i.e. metric properties. For the electromagnetic field, this has been stated many times [13, 26, 31], recently by Hehl et al. [15, 16, 28]. This is exactly our point of view. We intend here to consider the relation of this construction — generalized to any field — to the Ricci tensor, that was Schrödinger's favorite choice. It is the matter Lagrangian that is important when we intend to define a metric. Because it is quadratic in the derivatives of the fields, we have to use $\sqrt{-\det\ R_{ab}}\ \ d^4x$ itself as the invariant volume element, or alternatively we have to use fields that are densities of weight 1/2 ??????????[12]???????????.

To construct the matter part of the action within a local theory, we first recall that in GRT this is given through

$$S_2 = \int L_{\rm matter}[\Phi^A, \Phi^A{}_{;k}, g^{ik}] \sqrt{-\det\ g_{ab}}\ d^4x\,. \tag{4}$$

Our construction is however performed by using R_{ik} instead of g_{ik}. Then, we have two options. First, we can try actions with Lagrangians not explicitly containing the Ricci tensor, where the latter enters only the volume element,

$$S_3 = \int L_{\rm matter}[\Phi^A, \Phi^A{}_{;k}] \sqrt{-\det\ R_{ab}}\ d^4x\,. \tag{5}$$

Alternatively, we may consider a matter action similar to Eq. (4), but not necessarily implying the equality given by Eq. (3),

$$S_4 = \int L_{\rm matter}[\Phi^A, \Phi^A{}_{;k}, R_{ik}] \sqrt{-\det\ R_{ab}}\ d^4x\ \ . \tag{6}$$

Jakubiec and Kijowski have shown that the latter action can be transformed into GRT with a different set of non-gravitational fields. This helps with respect to the dynamical structure, but destroys the interpretation of the deliberately chosen fields [18, 19].

It is, of course, a drawback in the local action that matter has to exist *locally* in order to have a geometry defined. In the elementary vacuum, $\Phi_A \equiv 0$, the Euler–Lagrange equation might not exist or show singular behaviour. The matter in the surrounding universe only, like in Machian approaches, and not the purely local one, should be as necessary as sufficient to fix a geometry. However, a non–local Lagrangian will be the next step. First, we intend to evaluate a local action.

II. COVARIANT FIELD EQUATIONS

In this section, we intend to show the construction of covariant field equations derived from the action, eq.(5), in the case where the transformation properties of the field components Φ^A are not yet defined. We want to keep our formalism as general as possible, therefore we consider our basic matter field of the following form $\Phi^A \equiv [\Phi^{i_1\cdots i_{m_1}}{}_{k_1\cdots k_{n_1}}, \Phi^{i_1\cdots i_{m_2}}{}_{k_1\cdots k_{n_2}}, ...]$; that is, A represents field components of different fields with different transformation properties.

We assume, as usual, a local variational principle to get the Euler–Lagrange field equations in which L denotes the Lagrangian in the form $L = L[\Phi^A, \Phi^A{}_{;k}]$, to be distinguished from the form $L = L^*[\Phi^A, \Phi^A{}_{,k}, \Gamma^i{}_{kl}]$:

$$\frac{\partial L^*[\Phi^C,\Phi^C{}_{,l}]\sqrt{-\det\, R_{ab}}}{\partial\Phi^A} - \frac{\partial^2}{\partial x^k}\frac{L^*[\Phi^C,\Phi^C{}_{,l}]\sqrt{-\det\, R_{ab}}}{\partial\Phi^A{}_{,k}} = 0, \tag{7}$$

which are valid for a general tensor field, Φ^A, yet unspecified. Since we want to use covariant variables $L[\Phi^C,\Phi^C{}_{;l}]$ instead of $L^*[\Phi^C,\Phi^C{}_{,l}]$, it is more appropriate to write Eq. (7) in a covariant form. In order to do that, we define the covariant derivative through equation (1).

The change from partial to covariant derivatives implies that

$$\begin{aligned}\frac{\partial L^*[\Phi^C,\Phi^C_{,l}]}{\partial\Phi^A} &= \frac{\partial L[\Phi^C,\Phi^C_{;l}]}{\partial\Phi^A} + \frac{\partial L[\Phi^C,\Phi^C_{;l}]}{\partial\Phi^B{}_{;m}}\frac{\partial\Phi^B{}_{;m}}{\partial\Phi^A}\\ &= \frac{\partial L}{\partial\Phi^A} + \frac{\partial L}{\partial\Phi^B{}_{;m}}C^B{}_A{}^j{}_i\Gamma^i{}_{jm}\ , \end{aligned}\tag{8}$$

$$\frac{\partial L^*[\Phi^C,\Phi^C{}_{,l}]}{\partial\Phi^A{}_{,k}} = \frac{\partial L[\Phi^C,\Phi^C{}_{;l}]}{\partial\Phi^A{}_{;k}} \tag{9}$$

Then,

$$\begin{aligned}\frac{\mathrm{D}}{\partial x^l}\frac{\partial L}{\partial\Phi^A{}_{;k}} &= \frac{\partial}{\partial x^l}\frac{\partial L}{\partial\Phi^A{}_{;k}} + \Gamma^k{}_{jl}\frac{\partial L}{\partial\Phi^A{}_{;j}}\\ &\quad -C^B{}_A{}^n{}_m\Gamma^m{}_{nl}\frac{\partial L}{\partial\Phi^B{}_{;k}},\end{aligned}\tag{10}$$

where $\frac{\mathrm{D}}{\partial x^k}() \equiv ()_{;k}$, and

$$\frac{\mathrm{D}}{\partial x^k}\frac{\partial L}{\partial\Phi^A{}_{;k}} = \frac{\partial}{\partial x^k}\frac{\partial L}{\partial\Phi^A{}_{;k}} + \Gamma^k{}_{jk}\frac{\partial L}{\partial\Phi^A{}_{;j}} - C^B{}_A{}^n{}_m\Gamma^m{}_{nk}\frac{\partial L}{\partial\Phi^B{}_{;k}}\ . \tag{11}$$

The determinant of the Ricci tensor transforms as follows

$$\frac{\partial}{\partial x^k}\left(\ln\sqrt{-\det\, R_{ab}}\right) = \frac{\mathrm{D}}{\partial x^k}\left(\ln\sqrt{-\det\, R_{ab}}\right) + \Gamma^m{}_{mk}\ , \tag{12}$$

where we assumed that (in contrast to the notation in GRT) $R^{ij}R_{jk} = \delta^i_k$. Note that though the metric tensor also possesses this property, it is not necessarily implied a relation of the type given by Eq. (3). In fact, below we will see that in the presence of matter fields the Ricci and the metric tensors must be different. Combining the above formulas, we obtain the tensorial equation

$$\begin{aligned}&\frac{\partial L}{\partial\Phi^A} - \frac{\mathrm{D}}{\partial x^k}\frac{\partial L}{\partial\Phi^A{}_{;k}} - \left[\frac{\mathrm{D}}{\partial x^k}\ln\sqrt{-\det\, R_{ab}} + 2\Gamma^m{}_{[mk]}\right]\\ &\times\frac{\partial L}{\partial\Phi^A{}_{;k}} = 0.\end{aligned}\tag{13}$$

This is the covariant field equation, valid for a general matter field Φ^A.

Up to this point, an explicit dependence of L on R_{ik} was not involved. We now turn to the more general action, eq.(6). The equation for the affine connection,

$$\begin{aligned}&\frac{\partial L[\Phi^C,\Phi^C{}_{;l},R_{mn}]\sqrt{-\det\, R_{ab}}}{\partial\Gamma^a{}_{bc}}\\ &-\frac{\partial}{\partial x^k}\frac{\partial L[\Phi^C,\Phi^C_{;l},R_{mn}]\sqrt{-\det\, R_{ab}}}{\partial\Gamma^a{}_{bc,k}} = 0,\end{aligned}\tag{14}$$

is in this form neither tensorial nor covariant. We start from the definition of the Ricci tensor,

$$R_{ik} \equiv \Gamma^l{}_{il,k} - \Gamma^l{}_{ik,l} + \Gamma^l{}_{im}\Gamma^m{}_{lk} - \Gamma^l{}_{ik}\Gamma^m{}_{lm}\ . \tag{15}$$

Straightforward calculation yields

$$\begin{aligned}\frac{\partial R_{ik}}{\partial\Gamma^a{}_{bc}} &= \Gamma^r{}_{st}E_{rika}{}^{stbc}\ ,\end{aligned}\tag{16}$$

$$E_{rika}{}^{stbc} = \delta^c_r\delta^b_i\delta^t_k\delta^s_a + \delta^b_r\delta^s_i\delta^c_k\delta^t_a - \delta^b_r\delta^s_i\delta^t_k\delta^c_a - \delta^t_r\delta^b_i\delta^c_k\delta^s_a\ ,$$

$$\frac{\partial R_{ik}}{\partial\Gamma^a{}_{bc,d}} = D_{ika}{}^{bcd} = \delta^c_a\delta^b_i\delta^d_k - \delta^d_a\delta^b_i\delta^c_k\ . \tag{17}$$

Formally, the variational derivatives are

$$\begin{aligned}\frac{\delta[\sqrt{-\det\, R_{ab}}\,L]}{\delta[\Gamma^a{}_{bc}]} &= \frac{\partial\sqrt{-\det\, R_{ab}}\,L}{\partial\Gamma^a{}_{bc}} - \frac{\partial}{\partial x^d}\frac{\partial\sqrt{-\det\, R_{ab}}\,L}{\partial\Gamma^a{}_{bc,d}}\\ &= \boldsymbol{P}_B{}^cC^B{}_A{}^b{}_a\Phi^A\\ &+\ E_{rika}{}^{stbc}\Gamma^r{}_{st}G^{ik} - D_{ika}{}^{bcd}\frac{\partial}{\partial x^d}G^{ik}\ ,\end{aligned}\tag{18}$$

where we used the abbreviations

$$\boldsymbol{G}^{ik} \equiv \frac{\partial\sqrt{-\det\, R_{ab}}\,L}{\partial R_{ik}} \quad\text{and}\quad \boldsymbol{P}_B{}^c \equiv \sqrt{-\det\, R_{ab}}\,\frac{\partial L}{\partial\Phi^B{}_{;c}}\ . \tag{19}$$

We now solve the Euler–Lagrange equation for G^{ik} through use of the relation

$$D_{ika}{}^{bcd}\left(\delta^a_l\delta^m_b\delta^n_c - \frac{1}{3}\delta^a_c\delta^n_l\delta^m_b\right) = -\delta^d_l\delta^m_i\delta^n_k \tag{20}$$

and obtain

$$\frac{\partial}{\partial x^l}G^{mn} = -\sqrt{-\det\, R_{ab}}\,\frac{\partial L}{\partial \Phi^B{}_{;c}}C^B{}_A{}^b{}_a\Phi^A(\delta^a_l\delta^m_b\delta^n_c - \frac{1}{3}\delta^a_c\delta^n_l\delta^m_b)$$
$$- E_{rika}{}^{stbc}\Gamma^r{}_{st}G^{ik}(\delta^a_l\delta^m_b\delta^n_c - \frac{1}{3}\delta^a_c\delta^n_l\delta^m_b) \qquad (21)$$
$$= -\boldsymbol{P}_B{}^nC^B{}_A{}^m{}_l\Phi^A + \frac{1}{3}\delta^n_l\boldsymbol{P}_B{}^sC^B{}_A{}^m{}_s\Phi^A$$
$$- \Gamma^n{}_{ls}G^{ms} - \Gamma^m{}_{sl}G^{sn} + \Gamma^s{}_{ls}G^{mn} + \frac{1}{3}\delta^n_lG^{ms}(\Gamma^t{}_{ts} - \Gamma^t{}_{st}) \qquad (22)$$

There are two contractions,

$$\frac{\partial}{\partial x^n}\boldsymbol{G}^{nm} = -\boldsymbol{P}_B{}^mC^B{}_A{}^n{}_n\Phi^A + \frac{1}{3}\boldsymbol{P}_B{}^nC^B{}_A{}^m{}_n\Phi^A - \Gamma^m{}_{st}\boldsymbol{G}^{st} + \frac{1}{3}\boldsymbol{G}^{ms}(\Gamma^t{}_{ts} - \Gamma^t{}_{st}) \qquad (23)$$

and

$$\frac{\partial}{\partial x^n}\boldsymbol{G}^{mn} = \frac{1}{3}\boldsymbol{P}_B{}^nC^B{}_A{}^m{}_n\Phi^A - \Gamma^m{}_{st}\boldsymbol{G}^{st} + \frac{1}{3}\boldsymbol{G}^{ms}(\Gamma^t{}_{ts} - \Gamma^t{}_{st}) \qquad (24)$$

that is,

$$\frac{\partial}{\partial x^n}(\boldsymbol{G}^{mn} - \boldsymbol{G}^{nm}) = \boldsymbol{P}_B{}^mC^B{}_A{}^n{}_n\Phi^A\ . \qquad (25)$$

This is the generalisation of the known relation in Schrödinger's theory.

In our particular case, we obtain

$$\Gamma^b{}_{ka}R^{ck} + \Gamma^c{}_{ak}R^{kb} - \Gamma^k{}_{ak}R^{cb} - \Gamma^b{}_{kl}R^{lk}\delta^c_a + 2\frac{\partial \mathrm{ln}L}{\partial\Gamma^a{}_{bc}} = \left[\ln L\sqrt{-\det\, R_{ab}}\right]_{,k}\left[\delta^c_aR^{kb} - \delta^k_aR^{cb}\right] + \left[\delta^c_aR^{kb} - \delta^k_aR^{cb}\right]_{,k}. \qquad (26)$$

By contracting the indices b and c, and substituting that equation again into Eq. (26) implies that

$$R^{cb}{}_{,a} + \left[\ln\sqrt{-\det\, R_{ab}}\right]_{,a}R^{cb} + \left[\frac{2}{3}\Gamma^l{}_{[kl]}\delta^c_a - \Gamma^l{}_{al}\delta^c_k + \Gamma^c{}_{ak}\right]R^{kb} + \Gamma^b{}_{ka}R^{ck} = -2\left[\frac{\partial \mathrm{ln}L}{\partial\Gamma^a{}_{bc}} - \frac{1}{3}\frac{\partial \mathrm{ln}L}{\partial\Gamma^r{}_{bs}}\delta^r_s\delta^c_a\right]. \qquad (27)$$

For $\frac{\partial \mathrm{ln}L}{\partial\Gamma^a{}_{bc}}$, we obtain

$$\frac{\partial \mathrm{ln}L}{\partial\Gamma^a{}_{bc}} = \frac{\partial \mathrm{ln}L}{\partial\Phi^A{}_{;c}}C^A{}_B{}^b{}_a\Phi^B + \frac{\partial \mathrm{ln}L}{\partial R_{mn}}\frac{\partial R_{mn}}{\partial\Gamma^a{}_{bc}}. \qquad (28)$$

Schrödinger discovered that by defining a new affinity, ${}^*\Gamma^a{}_{bc} \equiv \Gamma^a{}_{bc} + \frac{2}{3}\delta^a_b\Gamma^l{}_{cl}$, equation (27) with L =const. reduces to $R^{cb}{}_{\overset{*}{;}a} \equiv R^{cb}{}_{,a} + {}^*\Gamma^c{}_{ka}R^{kb} + {}^*\Gamma^b{}_{ak}R^{ck} = 0$. In our case, these definitions imply that

$$R^{cb}{}_{\overset{*}{;}a} = -2\left[\frac{\partial \mathrm{ln}L}{\partial^*\Gamma^a{}_{bc}} - \frac{1}{3}\left[\frac{\partial \mathrm{ln}L}{\partial^*\Gamma^k{}_{kc}}\delta^b_a + \frac{\partial \mathrm{ln}L}{\partial^*\Gamma^k{}_{bk}}\delta^c_a\right]\right] + \left[\frac{\partial \mathrm{ln}L}{\partial^*\Gamma^a{}_{kl}} - \frac{1}{3}\left[\frac{\partial \mathrm{ln}L}{\partial^*\Gamma^k{}_{kl}}\delta^m_a + \frac{\partial \mathrm{ln}L}{\partial^*\Gamma^k{}_{mk}}\delta^l_a\right]\right]R_{ml}R^{cb} \qquad (29)$$

The introduction of matter fields ($L \neq$const.) avoids R^{cb} being parallel transported into itself by the star affinity;

the same holds for the Einstein affinity, see Ref. [40]. Then, the presence of matter fields preclude us to interprete the Ricci tensor as being the metric, see Eq. (3).

III. THE METRIC OF SPACE–TIME IN THE SHOCK-WAVE PICTURE

There are two kinds of ideal test objects that measure the geometry of space-time independent of their internal structure and constitution. This are the pole particles and the shock waves. Taken together, the propagation of shock waves (the bisectrices of wave equations) and the world-lines of pole particles yield the metrical structure [11]. Here, we consider field theory, hence we identify the metric, in particular the light-cone structure, through the propagation of shock waves. The observation of the propagation of (shock) waves defines the metric of the wave in question. In ordinary wave mechanics, the wave operator determines the shocks to propagate along its bisectrices. Each wave equation has its own causal cone when the wave operators differ in the highest order of derivatives. The principle of relativity requires that the propagation is the same for the different fields that one intends to include as fundamental, but this is a second question. In a construction like the action given by Eq. (5), the propagation of shock waves is given through substitution of

$$\Phi_{\rm shock} = \Phi_0 + \theta[z] z^2 \phi \tag{30}$$

for the fields Φ, where $z = z[x^k] = 0$ defines the shock hypersurface, Φ_0 and ϕ are at least C_2 in a neighborhood of the shock. The difference in the second-order derivatives of the two sides of the hypersurface is

$$\Phi_{,ik}{}^{+} - \Phi_{,ik}{}^{-} = \phi \; z_{,i} z_{,k} \tag{31}$$

On the shock front, the Euler-Lagrange equation requires [5]:

$$\frac{\partial^2 L}{\partial \Phi^A_{,i} \partial \Phi^B_{,k}} \phi^B z_{,i} z_{,k} = 0 \; . \tag{32}$$

In the case of only one scalar field, the result is trivially

$$g^{ik} \propto \frac{\partial^2 L}{\partial \Phi_{,i} \partial \Phi_{,k}} \tag{33}$$

In the case of more than one field component, we obtain a component-dependent propagation of the form

$$C_{AB}{}^{ik} \phi^B \; z_{,i} z_{,k} = 0 \, , \tag{34}$$

where $C_{AB}{}^{ik} \equiv \frac{\partial^2 L}{\partial \Phi^A_{,i} \partial \Phi^B_{,k}}$. This represents the eikonal equations that were already used by Lichnerowicz in determination of the effective metric in the Hermitean unified field theory, Ref. [20]. In GRT, we have the same form. However, local Lorentz invariance requires that this form determines a unique propagation cones at least for the fundamental free fields. Therefore, the GRT implies separability, i.e.,

$$C_{AB}{}^{ik} = a_{AB} g^{ik} \tag{35}$$

in order to obtain equal propagation cones for all field components [2, 5]. When we expect to get some definition of at least approximate metric properties from eq. (34), we have to look for a kind of approximate separability,

$$C_{AB}{}^{ik} = a_{AB} g^{ik} + \varepsilon_{AB}{}^{ik} \; , \quad \varepsilon \text{ small} \; . \tag{36}$$

Note that the coefficients $C_{AB}{}^{ik}$ depend **in general** on the construction of L, and not on the construction of the volume element $\sqrt{-\det R_{ab}}\, {\rm d}^4x$. The space–time Ricci curvature is irrelevant for the propagation of the shocks as long as it is not *explicitely* used in forming $L_{\rm matter}[\Phi^A, \Phi^A{}_{;k}, R_{ik}]$. However, explicit use implies higher order non-linearity, again.

IV. LOCAL ACTION INTEGRALS

With respect to our topic, there are three types of local action integrals, but none of them yields a a viable posteriori metric. Representative for the first type, i.e. action S_3, eq. (5), we assume a contravariant vector field Φ^k. When the Ricci tensor enters the action through the volume element only, we can construct actions like

$$S_3 = \int (\alpha \; \Phi^k_{;l} \Phi^l{}_{;k} + \beta \; \Phi^k_{;k} \Phi^l{}_{;l}) \sqrt{-\det R_{ab}} \; {\rm d}^4 x \tag{37}$$

We obtain

$$C_{ab}{}^{ik} = \frac{\partial^2 L}{\partial \Phi^a{}_{;i} \partial \Phi^b{}_{;k}} = 2 \; \alpha \; \delta^k_a \delta^i_b + 2 \; \beta \; \delta^i_a \delta^k_b \tag{38}$$

and, therefore,

$$\phi^b z_{,b} = 0. \tag{39}$$

This is a limitation only for the amplitude of the shock, and no limitation for its front. Again, the form of the volume element does not enter the shock condition. In a local theory, its construction cannot yield the metric of space–time.

It is not difficult to see that in a local theory any propagation depends on the local amplitudes of the interacting fields and not on the geometry of the shock fronts as long as the lightcones are not deliberately constructed through use of some second–order contravariant tensor field, i.e. an a priori metric. Such an a priori metric destroys our program, and cannot be its solution.

For the second type, we take Schrödinger's choice to go around the fatal result that the matter fields itself cannot locally determine a viable light cone, S_4, eq. (6). We still stick to a local construction and replace the ordinary metric with the Ricci tensor. This might not be the final construction because one expects, at least approximately, metricity of the connection [6], but we need only the shock approximation and the qualitative features of the field equations for our argument. We consider an action of the type given by Eq. (6) that is constructed using the methods of GRT or of the metric-affine theory followed by a substitution of R_{ik} for g_{ik} (the undifferentiated g_{ik}, not in the connection $\Gamma^a{}_{bc}$) The propagation of matter fields, of course, follows now the cone that is determined by R^{ik} as constructed. However, the field equation for the connection now yields a restricting condition for the decisive part of the energy-momentum tensor density,

$$\boldsymbol{T}^{ik} = \frac{\delta[L[g_{..}, \Phi^A, (\Phi^A{}_{,m} + C^A{}_B{}^b{}_a \Gamma^a{}_{bm}[g_{..}, g_{..}, .]\Phi^B)]\sqrt{-\det\, g_{ab}}\,]}{\delta[g_{ik}]}\,, \quad (40)$$

namely,

$$\boldsymbol{G}^{ik} = \frac{\partial(L[g_{..}, \Phi^A, \Phi^A{}_{;m}]\sqrt{-\det\, g_{ab}}\,)}{\partial g_{ik}}|_{\text{at } g_{..}=R_{..}}\,, \quad (41)$$

where the implicit dependence of $\Phi^A{}_{;m}$ on g_{ik} and its derivatives does not enter. We arrive at field equations for the connection, Eq. (21), that restrict the energy-momentum tensor density to kind of constant values, i.e., to peculiar, and not general, physical cases.

Third, we may use matter fields to construct a volume element in order to get field equations in space–times without curvature, too [1, 12]. In doing that, it is difficult not to introduce an a priori metric. Akama and Terazawa [1] hide it in the summation of their scalar fields, Gronwold et al [12] have it explicitly in their Lagrangians (see their section IV).

The construction of a metric through local non-gravitational fields has the consequence that a strong dependence of the metric on local perturbation must be expected. For instance, the metric components should be expected to be proportional to the local mass in zeroth order already. Therefore, we conclude that:

1. an a posteriori observation-based definition of a metric must rely on non-gravitational fields even in presence of a curved affine connection, and
2. its definition requires an explicitly non–local action for the non–gravitational fields.

Summarizing: Local theories of the type given by Eqs. (5) and (6) do not achieve a viable causal structure. In the former case, when the Ricci tensor only enters the volume element, the shock waves of the matter fields do not feel that metric and are not null surfaces as expected. In the latter case, when the Ricci tensor is deliberately substituted for the metric, the Schrödinger result of a covariantly constant Ricci tensor turns into a correspondingly constant matter tensor and excludes nearly all physical cases. It was our intention to show this in due generality.

When we construct action integrals with fields and connections alone we find field equations that exist only in the case when both Φ^A and R_{ik} are non-trivial. If there is no matter, the geometry cannot be measured and is free. If there is no curvature, the motion of matter is not defined. It is, of course, a drawback in the local action that matter has to exist locally in order to have a geometry defined. The geometry of space-time, however, is defined locally with matter only at large distance, as Mach observed.

V. CONCLUSION

After we have shown that local actions will not produce a posteriori metrics dynamically, we have to make some remarks about what to expect from non-local action integrals. Non-local interaction is constructed through at least twofold integration over space–time. The action, however, is a delicate point to be constructed properly. For instance, as long as the Lagrangian L can be expanded in a series of scalar functions at x with coefficients that are scalar functions at y, one of the integrations can formally performed and the result is again a local action. No new physics is found.

When we then try to implement terms like $\Psi^k[x]\Phi_k[y]$, we have to see that they are not scalars at all: $\Psi^k[x]$ is a vector for substitutions of x, and a scalar for substitutions of y. On the opposite, $\Phi_k[y]$ is a scalar for substitutions of x, and a vector for substitutions of y. The product can be made a scalar for both substitutions only when there exist a bi-tensor $\gamma_l{}^m[x, y]$ that depends on the two points, and transforms as a covariant vector when the x are substituted, and as a contravariant vector when y is substituted [37]. In this case, $\Psi^l[x]\gamma_l{}^m[x, y]\Phi_m[y]$ is a scalar and may be used in constructing Lagrangians. However, with the bi–tensor γ, we introduced a corresponding teleparallel connection [37], and the connection $\Gamma^a{}_{bc}$ is no more needed for the constuction. In addition, it is now equivalent to a tensor field of third order. Using only the connection constructed from the bitensor, we lose the Ricci tensor as equivalent of some metric: The curvature of a teleparallel connection vanishes, and the Schrödinger choice must be replaced by some other construction. In addition, it is important to note that the bi-tensor γ is mixed-variant. A covariant bitensor $\gamma_{ik}[x, y]$ is the generalization of the metric tensor $g_{ik}[x] = \lim_{y \to x} \gamma_{ik}[x, y]$. Teleparallel theories are discussed in connection as with string theory as with rotation in the universe [27, 29]. This is another story, of course.

The main problem is still that there is no convincing principle to conduct us. Nevertheless, it appears to be the only way to realize an a posteriori definition of the metric in a theory that does not use any metric explicitly besides the action itself.

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