ELECTROSTATICS AND CONFINEMENT IN EINSTEIN'S UNIFIED FIELD THEORY

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Einstein's unified field theory was devised already in 1925 for unifying gravitation and electromagnetism through the use of a nonsymmetric fundamental tensor g_{ik} and of a nonsymmetric affine connection Γ^i_{kl}; since 1945 it was intensely pursued by both Einstein and Schrödinger, who assumed it to be a complete field theory, not allowing for phenomenological source terms. However in 1955 a disappointed Erwin Schrödinger wrote: "It is a disconcerting situation that ten years endeavour of competent theorists has not yielded even a plausible glimpse of Coulomb's law." It occurred to us, through the study of exact solutions found in the meantime, that physically relevant results instead appear as soon as sources are allowed for, like it happens with the sources of gravity in the general relativity of 1915. We consider here the Hermitian version of the theory. Let

$$g_{ik} = g_{(ik)} + g_{[ik]}, \quad \Gamma^i_{kl} = \Gamma^i_{(kl)} + \Gamma^i_{[kl]}, \tag{1}$$

be both Hermitian, i.e. endowed with real symmetric and purely imaginary antisymmetric parts. We pose

$$g^{il}g_{kl} = \delta^i_k, \quad \mathbf{g}^{ik} = \sqrt{-g}g^{ik}, \tag{2}$$

where $g \equiv \det(g_{ik})$ is real. Then the original field equations without sources read:

$$g_{ik,l} - g_{nk}\Gamma^n_{il} - g_{in}\Gamma^n_{lk} = 0, \tag{3}$$

$$\mathbf{g}^{[is]}_{\ \ ,s} = 0, \tag{4}$$

$$R_{(ik)}(\Gamma) = 0, \tag{5}$$

$$R_{[[ik],l]}(\Gamma) = 0, \tag{6}$$

where $R_{ik}(\Gamma)$ is the usual Ricci tensor, Hermitian thanks to (3) and (4).

A class of solutions to these equations was found[1], that depend intrinsically on three coordinates. These solutions show that there is merit in appending sources at the right-hand sides of all the field equations, while preserving the Hermitian symmetry through the symmetrised Ricci tensor $\bar{R}_{ik}(\Gamma)$ of Borchsenius[2], that is equal to $R_{ik}(\Gamma)$ wherever the sources vanish. Works by Lichnerowicz on the Cauchy problem[3] and by Hély[4] show that the Bianchi identities look physically meaningful[5] when the metric is chosen to be a symmetric tensor s^{ik} such that

$$\sqrt{-s}s^{ik} \equiv \sqrt{-g}g^{(ik)}, \tag{7}$$

where $s^{il}s_{kl} = \delta^i_k$, and $s \equiv \det(s_{ik})$, and sources are appended at the right-hand sides of (3)-(6), in the form of a symmetric energy tensor T_{ik}, and of two conserved currents $4\pi\mathbf{j}^i = \mathbf{g}^{[is]}_{\ ,s}$ and $8\pi K_{ikl} = \bar{R}_{[[ik],l]}$. Then the Bianchi identities read:

$$\mathbf{T}^{ls}_{\ ;s} = \frac{1}{2}s^{lk}\left(\mathbf{j}^i\bar{R}_{[ki]} + K_{iks}\mathbf{g}^{[si]}\right). \tag{8}$$

The fundamental tensor for the general electrostatic solution[6] reads

$$g_{ik} = \begin{pmatrix} -1 & 0 & 0 & a \\ 0 & -1 & 0 & b \\ 0 & 0 & -1 & c \\ -a & -b & -c & d \end{pmatrix}, \text{ where:} \tag{9}$$

$$\begin{aligned} d &= 1 + a^2 + b^2 + c^2, \\ a = i\chi_{,x}, b &= i\chi_{,y}, c = i\chi_{,z}, \\ \chi_{,xx} + \chi_{,yy} &+ \chi_{,zz} = 0. \end{aligned} \tag{10}$$

The imaginary part $g_{[ik]}$ of the fundamental tensor looks like the general electrostatic solution of Maxwell's theory, because the "potential" χ must obey Laplace's equation, and

$$\mathbf{g}^{[is]}_{\ ,s} = 0\ , \quad g_{[[ik],l]} = 0,$$

happen to be satisfied. The squared interval reads:

$$\mathrm{d}s^2 = s_{ik}\mathrm{d}x^i\mathrm{d}x^k = -\sqrt{d}\left(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 - \mathrm{d}t^2\right) - \frac{1}{\sqrt{d}}(\mathrm{d}\chi)^2. \tag{11}$$

If only one point charge h is present in the "Bildraum" x, y, z, t, say, at the origin of the coordinates, the "potential" χ is

$$\chi = -\frac{h}{(x^2 + y^2 + z^2)^{1/2}}. \tag{12}$$

For this occurrence to happen, a net charge must appear at the right-hand side of the equation $\mathbf{g}^{[is]}_{\ ,s} = 0$. The surface $\mathrm{d}\chi = 0$, $d = 0$ is the inner border of the manifold. In the metric sense, this border is a point with a spherically symmetric neighbourhood.

A solution for n point charges at equilibrium can be built[6] by considering, in the "Bildraum", n closed surfaces possessing net charges, whose charge distribution reproduces the one occurring on n conductors at rest, due to their mutual induction. By changing the shapes and the positions of the surfaces in order to obtain that both $\mathrm{d}\chi = 0$ and $d = 0$ on each of them, one gets, in the metric sense, n point charges whose infinitesimal neighbourhood is spherically symmetric. When this occurs, even approximately, the position of the charges mimics the equilibrium positions prescribed by Coulomb's law[6].

Axially symmetric solutions allowing for n charged poles at the right-hand side of equation $R_{[[ik],l]} = 0$ are easily found[7]. In cylindrical coordinates $x^1 = r$, $x^2 = z$, $x^3 = \varphi$, $x^4 = t$ one writes the fundamental tensor:

$$g_{ik} = \begin{pmatrix} -1 & 0 & \delta & 0 \\ 0 & -1 & \varepsilon & 0 \\ -\delta & -\varepsilon & \zeta & \tau \\ 0 & 0 & -\tau & 1 \end{pmatrix}, \tag{13}$$

$$\begin{gathered} \zeta = -r^2 + \delta^2 + \varepsilon^2 - \tau^2, \\ \delta = ir^2\psi_{,r}, \varepsilon = ir^2\psi_{,z}, \tau = -ir^2\psi_{,t}, \\ \psi_{,rr} + \frac{\psi_{,r}}{r} + \psi_{,zz} - \psi_{,tt} = 0. \end{gathered} \tag{14}$$

The square of its line element reads:

$$\mathrm{d}s^2 = s_{ik}\mathrm{d}x^i\mathrm{d}x^k = \frac{\sqrt{-\zeta}}{r}\left(-\mathrm{d}r^2 - \mathrm{d}z^2 - r^2\mathrm{d}\varphi^2 + \mathrm{d}t^2\right) + \frac{r^3}{\sqrt{-\zeta}}(\mathrm{d}\psi)^2. \tag{15}$$

A static solution with n aligned poles built with K_{ikl} is found by requiring:

$$\psi = -\sum_{q=1}^{n} K_q \ln\frac{p_q + z - z_q}{r}, \quad p_q = [r^2 + (z - z_q)^2]^{1/2}; \tag{16}$$

K_q and z_q are constants. Let us consider the particular case when $n = 3$, and $K_1 = K_3 = K,\ K_2 = -K$, with $z_1 < z_2 < z_3$.

The charges are always pointlike in the metric sense; moreover, with the choice shown above, the metric happens to be spherically symmetric severally in the infinitesimal neighbourhood of each of the charges[7].

If chosen in this way, the three "magnetic" charges are always in equilibrium, like it would happen if they would interact mutually with forces independent of distance. The same conclusion was already drawn by Treder in 1957 from approximate calculations[8], while looking for electromagnetism in the theory. In 1980 Treder reinterpreted[9] his result as accounting for the confinement of quarks: in the Hermitian theory two "magnetic" poles with unlike signs are confined entities, because they are permanently bound by central forces of constant strength.

References

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