Jacobi's Principle and Hertz' definition of time

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in: Pronin,P., Sardanashvily,G. (eds.): Gravity, Particles and Space-Time,
World Scientific, Singapore 1996, pp.207-215

This article should remind the interest which D.D.Ivanenko always had in the fundamental questions of Mach's ideas for founding the physics of inertia. Even today, we have no generally accepted idea yet how to quantify the general demand for a theory, in which the existence and not only the amount of inertia of a body is determined by the configuration of the surrounding universe. The actual discussion centers around the problem of introducing time in theoretical constructions without time, and this paper shall be a contribution to this dicussion.

As in Machian mechanics as in the construction of a general-relativistic realization of a Machian theory as in quantum cosmology the question of the contruction of a suitable time variable has to be considered separately. Here, we intend to outlay the approaches of Jacobi and Hertz, apply them to Machian mechanics, and discuss them in the context of the superspace of canonical GRT.

Jacobi demonstrated how to separate explicitely the time from a mechanical problem not explicitely depending on time. If the lagrangian $L[q, \dot{q}, t]$ in the variational principle

$$\delta \int L[q, \dot{q}, t] \mathrm{d}t = 0$$

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with the Euler-Lagrange equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = 0 \ . \tag{1}$$

does not depend explicitely on time, i.e., if a energy is conserved,

$$\frac{\partial L}{\partial t} = 0 \quad \Longrightarrow \quad \sum_i \dot{q}^i \frac{\partial L}{\partial \dot{q}^i} - L = \mathrm{const} \ ,$$

the solutions can be shifted in time. A class of solutions differing just by such a shift defines a unique projection onto the configuration space, the same path. The bundle of pathes in the configuration space will be a bundle of geodesics, if the Lagrangian contains the velocities in a homogeneous quadratic term, i.e. if

$$\frac{\partial L}{\partial \dot{q}^i} = m_{ik}[q]\frac{\mathrm{d}q^i}{\mathrm{d}t} \Longrightarrow L = \frac{1}{2}\sum_{ik} m_{ik}[q]\frac{\mathrm{d}q^i}{\mathrm{d}t}\frac{\mathrm{d}q^k}{\mathrm{d}t} - V[q] \ .$$

The Euler-Lagrange equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}(m_{ik}[q]\frac{\mathrm{d}q^k}{\mathrm{d}t}) - \frac{1}{2}\frac{\partial m_{ab}[q]}{\partial q^i}\frac{\mathrm{d}q^a}{\mathrm{d}t}\frac{\mathrm{d}q^b}{\mathrm{d}t} + \frac{\partial V[q]}{\partial q^i} = 0 \ , \tag{2}$$

Are to be transformed in a geodesic equation by trying a metric [20, 23]

$$\mathrm{d}s^2 = n^2[q]m_{ik}[q]\mathrm{d}q^i\mathrm{d}q^k$$

of the configuration space, and we find by use of the energy conservation

$$\frac{1}{2}\sum_{ik} m_{ik}\frac{\mathrm{d}q^i}{\mathrm{d}t}\frac{\mathrm{d}q^k}{\mathrm{d}t} = E - V[q]$$

the only solution

$$n^2[q] = C\cdot(E - V[q]) \ .$$

C is an integration constant, defining the length unit on the geodesics.

The solutions of the Euler-Lagrange equations for a fixed energy E are geodesics to the metric

$$\mathrm{d}s^2 = (E - V[q])m_{ik}\mathrm{d}q^i\mathrm{d}q^k$$

in configuration space. Each value of the energy defines an own metric. Time has become an “ignorable” or “kinasthenic” variable. Its flow is a secondary question, answered by the known formula

$$\mathrm{d}t = \frac{\mathrm{d}s}{E - V[q]} = \sqrt{\frac{m_{ik}\mathrm{d}q^i\mathrm{d}q^k}{E - V[q]}}$$

for ephemeris time [4, 9, 14, 19, 22]. The main requirements of this Jacobi's construction are the conservation of energy $\frac{\partial}{\partial t}L = 0$ and the independence of the potential U of the velocities $\dot{\vec{r}}_A$: $\frac{\partial}{\partial \dot{q}}V = 0$. Jacobi's definition of time introduces the experimental time for celestial mechanics. This definition of time coincides with Lange's inertial time in case of force-free motion. Jacobi's principle contains newtonian mechanics if and only if the total energy is conserved. Time-independent mechanics reflect time-independent interactions and time-independent conditions. The path of the system in configuration space maps its history together with the flow of time.

Traditionally, the Lagrangian of a system of point particles is constructed as the difference of a kinetic energy with masses independent of configuration and a potential energy of pairwise interaction, depending only on position:

$$L = \frac{1}{2}\sum_A m_A \dot{\mathbf{r}}_A^2 - \frac{1}{2}\sum_{AB} U[\mathbf{r}_A, \mathbf{r}_B] \ .$$

This leads to metrics in configuration space, which are independent on configuration up to the scalar factor $E - V[q]$. In the case of inertia-free or relational mechanics, the traditional kinetic energy is replaced by a velocity-dependent interaction potential. The construction of the metric in the configuration space is not affected, but leads to a configuration-dependent expression, which cannot be split into a configuration-independent tensor and a scalar function of configuration. This leads immediately to the question of defining the split of the metric into the potential factor $\sqrt{E - V[q]}$ and the kinetic factor $m_{ik}[q]\mathrm{d}q^i\mathrm{d}q^k$. To consider the possible splits of the metric of configuration space into potential and kinetic factor we begin with the geodesic equation

$$\frac{\mathrm{d}}{\mathrm{d}s}(g_{ik[q]}\frac{\mathrm{d}q^k}{\mathrm{d}s}) - \frac{1}{2}\frac{\partial}{\partial q^i}(g_{ab}[q])\frac{\mathrm{d}q^a}{\mathrm{d}s}\frac{\mathrm{d}q^b}{\mathrm{d}s} = 0 \ . \tag{3}$$

We restrict the time lapse to be a scalar function of the arc length, $\mathrm{d}t = c^{-1}[q]\mathrm{d}s$. On the other hand, the expected expression for the kinetic energy should be constructed by some $T = \frac{1}{2}m[q](\frac{\mathrm{d}s}{\mathrm{d}t})^2$. We substitute equation (3) and compare with equation (1). We get

$$m[q]c[q] = 1 \ , \quad c[q] = 2(E - V[q]) \ ,$$

and that is all.

If the metric tensor $g_{ik}[q]$ in configuration space is given, we get for any positive function $n^2[q] = E - V[q]$ a mass tensor $m_{ik}[q] = (2(E - V[q]))^{-1}g_{ik}$ with the property: The variation problem to the lagrangian $L = \frac{1}{2}m_{ik}[q]\dot{q}^i\dot{q}^k - V[q]$ and the total energy E leads to orbits in configuration space, which are geodesics with respect to the metric $\mathrm{d}s^2 = g_{ik}[q]\mathrm{d}q^i\mathrm{d}q^k$. At this point, we are free to choose the potential factor $n^2[q]$, or the time flow $\mathrm{d}t = n[q]\mathrm{d}s$ respectively. The only intrinsically defined choice is $n = 1$, the time being the arc length in configuration space itself. This is a typical Machian property, although Mach did not see this role of Jacobi's formulation of the principle of least action. Only H.Hertz, in his "forceless mechanics" (1894) proposed the time-difference $t - t_0$ to be given by the length of the "straightest distance" S in the configuration-space:

$$(t - t_0)^2 = \frac{mS^2}{2E} \tag{4}$$

where m means the total mass and E the energy constraint of a (closed) system, and

$$m(\delta S)^2 = \sum_A m_A\ \delta\mathbf{x}_A^2\ .$$

Hertz pointed out that his integral definition of time turns over to Jacobi's (1842) differential definition

$$\mathrm{d}t^2 = \frac{\sum_B m_B \mathrm{d}s_B^2}{2(E - U)} \tag{5}$$

in the case of holonomic contraints and potential energy $U(x^i)$. For forceless motions without constraints Hertz' definition of time gives Lange's (1886) "inertial time" [15]. Stimulated by Mach's "Mechanik" H.Hertz claimed equation (16) to be the dynamical definition of the time t and Jacobi's differential equation (17) as the definition of differential time in the case of holonomic constraints. – Equation (17) shows that any space coordinate can replace the time t in the equations of motion.

In the special case of the existence of coordinates which allow a split

$$g_{ik} = n^2[q]m_{ik}$$

with configuration-independent mass tensor m_{ik}, there is only one choice of $n^2[q]$, or $\mathrm{d}t/\mathrm{d}s$ respectively, which declares just this mass tensor to form the kinetic energy in the traditional way, i.e.

$$T = \frac{1}{2}\sum_{ik} m_{ik}\frac{\mathrm{d}q^i}{\mathrm{d}t}\frac{\mathrm{d}q^k}{\mathrm{d}t}\ .$$

Up to free constant factor,

$$n^2[q] = 2(E - V[q])\ .$$

In the interpretation of inertia-free, or relational, mechanics, a configuration-independent mass tensor m_{ik} is an absolute, i.e. anti-Machian, element of the theory [23].

Einstein pointed out that Mach's principle involves the substitution of the space-time V_4 by the $3N$–dimensional configuration spaces of Lagrange and Hertz. Only the distances $(x^i_A - x^i_B)$ can have a physical interpretation if only the relative accelerations induce resistance (inertial forces). However, in order to implement that concept, General Relativity has to start with Levi-Civita's "absolute parallelism". Einstein remarks in his "Last lecture": "Recognized that possibility to avoid inertial system depend on existence of a Γ field that described parallelism in the infinitesimally small." The substitution of inertial systems by Levi-Civita's teleparallelism is also the point in Einstein's discourse on Mach's principle in his "Relativistic theory of the Non-Symmetric Field".

In the 5. appendix to the last edition of [5] Einstein considered "Relativity and the problem of space" with respect to Mach's principle and Descartes' philosophy of space. According to Einstein, Descartes' equivalence of space and matter is the solution of Mach's problem. Space and matter are given by the same entities, i.e. the affine connections ${\Gamma^\alpha}_{\mu\nu}$ of Levi-Civitá. In the "relativistic theory of the non-symmetric field" Einstein's affine tensors ${U^\alpha}_{\mu\nu} = {\Gamma^\alpha}_{\mu\nu} - {\delta^\alpha}_\nu {\Gamma^\lambda}_{\mu\lambda}$ are the unified gravitational and matter fields. The Riemann case with index symmetry ${\Gamma^\alpha}_{\mu\nu} = {\Gamma^\alpha}_{\nu\mu}$ represents purely gravitational fields.

In the purely affine field theory (Schrödinger 1950) the ${\Gamma^\alpha}_{\mu\nu}$ (repsectively the ${U^\alpha}_{\mu\nu}$) are the only field coordinates. The expressions $\delta^\alpha_\lambda R^{\mu\nu}$ are the canonically conjugated momenta [5, 24]. The Lagrangian is the Einstein-Schrödinger density (Einstein 1925, Schrödinger 1950):

$$\mathcal{H} = \frac{2}{\lambda}\sqrt{-\det(R_{\mu\nu})}\ , \quad \lambda > 0$$

with the field equations

$$\frac{\delta \mathcal{H}}{\delta R_{\mu\nu}} \frac{\delta R_{\mu\nu}}{\delta {\Gamma^\alpha}_{\beta\gamma}} = 0\ .$$

The Hamiltonian density to the Lagrangian $\mathcal{H}$ does not contain a separate interaction potential, but only some kind of kinetic energy density. If we interpret the equivalence classes of R_{ik} as the points of a superspace Σ, the Hamiltonian metric is

$$\mathrm{d}\sigma^2 = \frac{1}{2}(R^{im}R^{kl} + R^{il}R^{km} - R^{ik}R^{lm})\mathrm{d}R_{ik}\mathrm{d}R_{lm}$$

The Hertz principle of the now straightest path could be of the form

$$\delta \int \mathrm{d}\sigma$$

complementary to the Jacobi principle of shortest path. The relativistic space-time should be interpreted as the straightest path in the momentum superspace of the $\{R_{ik}\}$.

According to the definition of the Ricci tensor,

$$\begin{aligned} R_{\mu\nu} &= -\Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\mu\alpha,\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^{\beta}_{\alpha\beta} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\alpha\nu} \\ &= -U^{\alpha}_{\mu\nu,\alpha} + U^{\alpha}_{\mu\beta}U^{\beta}_{\alpha\nu} - \frac{1}{3}U^{\alpha}_{\mu\alpha}U^{\beta}_{\beta\nu} \ , \end{aligned}$$

the Hamiltonian superspace Σ is a subspace of the superspace $\Sigma^* : \{U^i_{kl}\}$ with the U^i_{kl} as coordinates. The subspace Σ is defined by the conditions

$$R_{ik} = -U^{\alpha}_{ik,\alpha} + U^{\alpha}_{i\beta}U^{\beta}_{\alpha k} - \frac{1}{3}U^{\alpha}_{i\alpha}U^{\beta}_{\beta k} \ , \tag{6}$$

The straightest path in Σ is the straightest path in Σ^* under the constraint, eq.(6). In the spirit of Hertz' dynamics, we see no interaction potential, but only constraints.

In the superspace $\mathcal{S} : \{g_{ab}\}$ of General Relativity the time-free Jacobi principle should be found. On the other hand, a conservation theorem for the energy can be expected only in the case of the existence of a timelike Killing vector field, i.e. if the metric tensor can be transformed into a coordinate system where it does not depend on the time coordinate in question. One could argue that the pathes in superspace, corresponding to a Jacobi principle, do not correspond to a covariant world because we have a coordinate system distinguished by time-independence.

The canonical formalism of General Relativity yields the Wheeler-DeWitt equation, which is the equivalent for the Schrödinger equation in quantum mechanics, and to the Hamilton-Jacobi equation in classical mechanics, respectively. The transversal trajectories of the solutions of the Wheeler-DeWittt equation are pathes in superspace and do not contain time. Time is found by some procedure of combining the spaces of the path to a four-dimensional world. To this end, one should try an analogue to the Jacobi principle. This has been done by [1]. However, the action integral of general relativity,

$$S = \int \mathrm{d}\lambda \int \mathrm{d}^3x \sqrt{{}^3R \ G^{abcd} \ (\frac{\partial g_{ab}}{\partial \lambda} - N_{a|b})(\frac{\partial g_{cd}}{\partial \lambda} - N_{c|d})} \ ,$$

is not exactly the generaliziation of Jabobi,

$$S = \int \mathrm{d}\lambda \sqrt{\int \mathrm{d}^3 x\, ^3R\; G^{abcd}\; (\frac{\partial g_{ab}}{\partial \lambda} - N_{a|b})(\frac{\partial g_{cd}}{\partial \lambda} - N_{c|d})}\ .$$

In our interpretation, this is due to the fact, that in ordinary mechanics we can argue with a mass tensor reducible to an absolute one (the masses in cartesian coordinates of space), not depending on dynamics. In GRT, the mass tensor given by the metric in superspace,

$$\mathrm{d}s_0^2 = G^{iklm}\mathrm{d}g_{ik}\mathrm{d}g_{lm} = \frac{1}{2}(g^{il}g^{km} + g^{im}g^{kl} - g^{ik}g^{lm})\mathrm{d}g_{ik}\mathrm{d}g_{lm} \tag{7}$$

is totally determined by the variable g_{ik} itself. The superspace metric is a constuct of the actual configuration and does not contain absolute metric quantities. This is true, of course, only in the traditional interpretation. An absolute element, contained in the construction of equation (7), is the Levi-Cività symbol ϵ^{ikl} necessary to find the inverse g^{ab} of the metric g_{ab}. One could interpret this fact as a hint, that the affine invariance represented by the abstract metric ϵ^{ikl} plays a more fundamental role than local Lorentz invariance represented by the metric $g_{\mu\nu}$. This is the point of view considered in [16, 2].

A Jacobi principle for superspace should have the form

$$\delta \int \sqrt{A[g_{ab}]\mathrm{d}s_0^2} = 0\ .$$

We know, that the choice

$$A[g_{ab}] = \ ^3R$$

yields a variational principle near to that of GRT. We could hope for a selection criterion, which choses out of the many possible time flows just one, which hopefully transforms the path in superspace into a covariant fourdimensional world. It is to be checked, if such a selection principle exists and makes the theory unique.

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