INTERPRETING SOLUTIONS WITH NONTRIVIAL KILLING GROUPS IN GENERAL RELATIVITY

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ABSTRACT. General relativity is reconsidered by starting from the unquestionable interpretation of special relativity, which (Klein 1910) is the theory of the invariants of the metric under the Poincaré group of collineations. This invariance property is physical and different from coordinate properties. Coordinates are physically empty (Kretschmann 1917) if not specified by physics, and one shall look for physics again through the invariance group of the metric. To find the invariance group for the metric, the Lie motion ("Mitschleppen" in his words) is ideal for this task both in special and in general relativity. For a general solution of the latter the invariance group is nil, and general relativity behaves as an absolute theory, but when curvature vanishes the invariance group is the group of infinitesimal Poincaré motions of special relativity. Solutions of general relativity exist with invariance groups intermediate between the previously mentioned extremes. The Killing group properties of the static solutions of general relativity were investigated by Ehlers and Kundt (1964). The particular case of Schwarzschild's solution is examined, and the original choice of the manifold done by Schwarzschild in 1916 is shown to derive invariantly from the uniqueness of the timelike, hypersurface-orthogonal Killing vector of that solution.

1. INTRODUCTION

The year 2010 lies, in physics, midway between two centenaries. At the time of writing, five years have elapsed since the centenary of the Einstein year 1905, with the onset of special relativity theory [1], and five years too separate us from the centenary of the discovery of the general theory of relativity [2], [3]. Both theories are in present times rightly studied and considered like pillars of physical knowledge. It is quite surprising to remark, then, that the mutual relationship between the two theories is still clouded by some fog that needs to be completely dissipated. This happens because, while the physical interpretation of the special theory of relativity is unique and clear since the writing, in 1910, of the well known sentence by

Felix Klein[2], the group theoretical interpretation of general relativity theory is still in want of further understanding of its physical meaning along the path set by the very early remarks of Kretschmann [6] and of Noether [7] respectively. This problematic situation becomes evident whenever solutions of that theory are encountered, that happen to possess a nontrivial Killing structure. Like we shall see in the following, this is a frequently overlooked, but by no means minor issue. In fact, for ease of computation, nearly all the solutions to the field equations of general relativity studied up to now by the relativists are endowed with some symmetry, i.e. their metric manifolds possess nontrivial Killing groups of invariance. Whether and how the occurrence of these nontrivial physical structures influences the physical interpretation of particular solutions will be the subject of the present investigation. The weight of the argument is best understood if one takes one step back, and starts the discussion from the nontrivial Killing structure associated with the invariance of the Minkowski metric manifold under the group of collineations mentioned by Klein, whose physical content was clear in 1910 and is still unquestionable today.

Although it is not immediately prominent, the discussion of the Killing structure is an extension of the good old principle that rest can be defined only with respect to some other object - to some other body in mechanics, to some other object in general. In classical mechanics, a body can be at rest with respect to itself, then it is inertially moving, no acceleration involved. It can be temporarily or permanently at rest with respect to some other body with no such condition. In Special Relativity, being at rest the one to the other requires for both inertial, acceleration-free motion because of the lack of absolute simultaneity. In GRT, the two notions of being at rest with repect to one-self and being at rest to another body fall apart. To be at rest with respect to one-self remains inertial motion. To be at rest with respect to another body implies to be at rest with respect to the structures of the gravitational field produced by this second body. A body is at rest in a gravitational field when the field does not change along its world-line. When such a world-line exists, it is a Killing trajectory. As we shall see, due to the more complicated structure of a metric field, the field itself must reflect the inertial state of its source. In the most general case, when the Killing group is trivial, the events of space-time may be intrinsically characterized through the different invariants to be constructed with the Riemann tensor, and any world-line passes through varying environments so that it can never be called to be at rest. The gravitational field must allow a congruence of timelike world lines, along which the field does not change. A particle moving on

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[2]"Was die modernen Physiker *Relativitätstheorie* nennen, ist die Invariantentheorie des vierdimensionalen Raum-Zeit-Gebietes, x, y, z, t (der Minkowskischen "Welt") gegenüber einer bestimmten Gruppe von Kollineationen, eben der "Lorentzgruppe"[4],[5]. English translation: "What the modern physicists call *theory of relativity* is the theory of the invariants of the space-time region, x, y, z, t (the Minkowski "world") with respect to a given group of collineations, namely the "Lorentz group".

such a world line can then be called at rest with respect to the field. In other words, a rest can be defined in a manifold with a timelike Killing congruence. These fields will be studied in section 5 . Due to the existence of the gravitational field, being at rest is no more an acceleration-free state. To be at rest in a gravitational field implies to be accelerated all the time, and the acceleration is the curvature of the Killing trajectories. When the Killing congruence is uniquely defined, the curvature of its lines yields again a measurable scalar quantity, as we shall discuss in section 5.

2. The Minkowski metric in curvilinear coordinates

The world-lines of free motion and all the other trajectories of the subgroup of translations of the Poincaré group form a set of straight lines, which is used to introduce the "Galilean coordinates"[8] we are used to. The metric expresses the full Poincaré group, and obtains the known form in Galilean coordinates. In GRT, the world lines of free motion do not form a set of straight lines any more, and the invariance group may be reduced to even the trivial group. Hence, no simplifiying coordinates exist.

No doubt, if the Minkowski "world" is described by availing of "Galilean coordinates"[8] x, y, z, t, with respect to which the metric g_{ik} reads

$$(2.1) \qquad g_{ik} = \eta_{ik} = \mathrm{diag}(-1, -1, -1, 1),$$

a great simplification occurs. When this representation is adopted, the coordinates x^i are not just labels for identifying events. Due to this particular form of η_{ik}, Galilean coordinates have a direct metric reading, i.e. to each particular system of coordinates a physically admissible, inertial reference frame, to be built with rods, clocks and light signals, is directly associated in one-to-one correspondence. Moreover, when this representation is adopted, one recognizes that the Poincaré group of transformations, besides being endowed with direct physical meaning, is the group of invariance of η_{ik}. The invariance of η_{ik} under the Poincaré group constitutes what Klein and later Kretschmann once called the physically meaningful "relativity postulate" of the original theory of relativity of 1905. Although Galilean coordinates are adopted for ease of representation, it is clear that the "relativity postulate", i.e. the group of invariance, is physical and coordinate independent. Therefore other representations, that do away from Galilean coordinates, could be availed upon as well.

To ease the comparison of special relativity with the general relativity of 1915, where the adoption of general curvilinear coordinates, with the associated group of covariance, in keeping with the fundamental work by Ricci and Levi-Civita [9], is *de rigueur*, it is necessary to describe the Minkowski metric manifold in general curvilinear coordinates too. In this way the duplicity characteristic of the Galilean coordinates disappears. In the Minkowski manifold curvilinear coordinates are just labels, devoid of physical meaning beyond the mere topological identification of the events. In fact the only

physical restriction on curvilinear coordinates, needed for preserving the individuality of the single event, is just that any transformation between two such systems of coordinates needs to be one-to-one. The elements of the abstract Poincaré group have in general no global representation through a coordinate transformation occurring between two arbitrarily given systems of curvilinear coordinates x^i and x'^i. Since the existence of the elements of the Poincaré group, meant in the abstract sense, does not depend on the choice of the coordinates, it is fundamental to learn what subgroup of the abstract Poincaré group, if any, can find a mathematically affordable representation with respect to a general system of coordinates. If this question is positively answered, studying the symmetries of a pseudo-Riemannian manifold with a unique mathematical formulation of general character that applies both whether the curvature tensor R_{iklm} of the metric manifold is vanishing or not will become a well defined problem.

It is clear that curvilinear coordinates are unsuitable in general for providing a global account of the symmetry properties of a curved manifold. If the curvature tensor R_{iklm} is nonvanishing one shall restrict the study of these symmetries to an infinitesimal neighbourhood of a given event. Happily enough, in this limit the powerful mathematical tool of Lie's infinitesimal "dragging along" ("Mitschleppen" [10]) of a metric field can be used [8].

Let us consider a pseudo Riemannian manifold equipped with two curvilinear coordinate systems x'^i and x^i ($i = 1, .., 4$) such that

$$(2.2) \qquad x'^i = x^i + \xi^i,$$

where ξ^i is an infinitesimal four-vector. Under this infinitesimal coordinate transformation, the components of the metric tensor g'^{ik} in terms of g^{ik} read

$$(2.3) \qquad g'^{ik}(x'^p) = \frac{\partial x'^i}{\partial x^l} \frac{\partial x'^k}{\partial x^m} g^{lm}(x^p) \approx g^{ik}(x^p) + g^{im} \frac{\partial \xi^k}{\partial x^m} + g^{km} \frac{\partial \xi^i}{\partial x^m}.$$

The quantities in the first and in the last term of (2.3) are calculated at the same event (apart from higher order infinitesimals). We desire instead to compare the quantities g'^{ik} and g^{ik} calculated for the same coordinate value, i.e. evaluated at neighbouring events separated by the infinitesimal vector ξ^i. To this end, let us expand $g'^{ik}(x^p + \xi^p)$ in Taylor's series in powers of ξ^p. By neglecting higher order infinitesimal terms, we can also substitute g^{ik} for g'^{ik} in the term containing ξ^i of the expansion truncated at the first order term, and we find:

$$(2.4) \qquad g'^{ik}(x^p) = g^{ik}(x^p) + g^{im} \frac{\partial \xi^k}{\partial x^m} + g^{km} \frac{\partial \xi^i}{\partial x^m} - \frac{\partial g^{ik}}{\partial x^m} \xi^m.$$

But the difference $\delta g^{ik}(x^p) = g'^{ik}(x^p) - g^{ik}(x^p)$ has tensorial character and can be rewritten as

$$(2.5) \qquad \delta g^{ik}(x^p) = \xi^{i;k} + \xi^{k;i}$$

in terms of the contravariant derivatives of ξ^i. When

$$(2.6) \qquad \xi^{i;k} + \xi^{k;i} = 0$$

$\delta g^{ik}(x^p) = 0$, and the metric tensor g^{ik} goes into itself under Lie's "Mitschleppen"[10] along ξ^i. An infinitesimal Killing vector is defined as a four-vector ξ^i that fulfills (2.6). We assume for instance that, at a given event, n infinitesimal Killing vectors $_a\xi^i, a = 1, .., n$ exist. They define the infinitesimal "Mitschleppen" group of rank n, against which the metric g_{ik} remains invariant. In another instance, let us consider the solutions of Eqs. (2.6) that hold when $R_{iklm} = 0$. Its vectors ξ^i define the elements of the infinitesimal Poincaré group. Eqs. (2.6) thereby provide the sought-for unique mathematical description of the local symmetries for both the special and the general theory of relativity through the corresponding Killing groups.

3. Kretschmann's objection to Einstein's interpretation of general relativity

Despite the warning implicit in Klein's ironic sentence of 1910 [4], when, at the end of the year 1915, both Einstein and Hilbert arrived at the field equations of general relativity, both of them thought that their fundamental achievement entailed, inter alia, the realisation of a theory of gravitation whose underlying group was the group of general coordinate transformations. At variance with Hilbert's standpoint, that the adoption of general coordinates was per se a great advance in physics, due to the extraordinary achievement thereby obtained in the mathematical structure of the theory, in Einstein's original idea the newly acquired group-theoretical property of general covariance was believed to be an essential one from a physical point of view. According to Einstein's original conception of general relativity [11], giving up the Galilean coordinates and the a priori Minkowski metric η_{ik} and admitting general, curvilinear coordinates might allow, on physical grounds, the introduction of reference frames that do away from the arbitrary singling out of the inertial frames as the only admissible ones. Since, according to the early version of the equivalence principle, gravity and acceleration of a test particle had to be identified locally at any given event, in the newborn theory of gravitation curvilinear coordinates should be introduced not just for availing of the convenient mathematical tools introduced by Ricci and Levi-Civita [9], but due to a cogent physical reason in the first place. It is mathematically exhibited by the shifty role of inertial and of gravitational forces, identified as the two nontensorial addenda that appear in, say, the equation of geodesic motion

$$(3.1) \qquad \frac{\mathrm{D}^2 x^i}{\mathrm{d}s^2} \equiv \frac{\mathrm{d}^2 x^i}{\mathrm{d}s^2} + \Gamma^i_{kl} \frac{\mathrm{d}x^k}{\mathrm{d}s} \frac{\mathrm{d}x^l}{\mathrm{d}s} = 0,$$

and were interpreted [11] by Einstein just like acceleration and gravitation respectively. In this way, the vanishing of acceleration or the vanishing of gravitation at a given event could be produced in principle through suitably chosen coordinate transformations belonging to the group of general coordinate transformations. In the same year 1915, however, Erich Kretschmann had published in *Annalen der Physik* a long article [12], entitled "Über

die prinzipielle Bestimmbarkeit der berechtigten Bezugssysteme beliebiger Relativitätstheorien" (On the fact that, in any relativity theory, legitimate systems of reference can be determined in principle), in which an accurate analysis of the relation between observation and mathematical structure in a theory possessing a generic postulate of relativity is developed. No wonder then, if two years later, with the paper [6] entitled "Über den physikalischen Sinn der Relativitätspostulate; A. Einsteins neue und seine ursprüngliche Relativitätstheorie" (On the physical meaning of postulates of relativity; A.Einstein's new and his original theory of relativity), the same author produced an analysis of the relation between the "special" and the "general" theory of relativity that defied the previously quoted group-theoretical assessment by Einstein, and proposed an alternative of his own, whose objection was entirely in keeping with Klein's remarks. The validity in principle of Kretschmann's objection was soon acknowledged by Einstein himself [13]. Acceptance was allotted henceforth to Kretschmann's way of assessing the very meaning of "relativity". In keeping with Kretschmann and Klein, the "relativity content" of a given theory should not be ascertained through the group of covariance allowed by the particular expression adopted for writing the equations of that theory in terms of certain coordinates. It should be assessed through its group of invariance, meant to be "a physical property of the system". As learned in the long time elapsed since the publication of Kretschmann's paper of 1917, in a Riemannian metric manifold the group of invariance of the metric is directly inscribed by the Killing vectors in the intrinsic, geometric structure of a manifold.

4. Interpretation of solutions with nontrivial Killing groups in general relativity

In keeping with Kretschmann's objection, curvilinear coordinates used to describe a certain solution of general relativity are physically vacuous, because the field equations of any theory could be written with such coordinates, while the invariant properties of the metric, accounted for by the associated Killing group, convey information on the physical content of the solution under question. The long known relation between invariance and true conservation of physical quantities, first investigated by Noether [7], is another proof of the validity of the latter assertion.

The seemingly obvious objection raised by Kretschmann has far-reaching consequences. First of all, while the Killing group of the metric of special relativity is just the Poincaré group in the limit case of infinitesimal motions, for a general solution of the field equations of general relativity the Killing group reduces to the identity, i.e. general relativity, despite its very name, in that case behaves indeed like an absolute theory. Moreover in general relativity particular solutions exist too, whose Killing group happens to be intermediate between the Poincaré group for infinitesimal motions, that

prevails when $R_{iklm} = 0$, and the trivial group that holds when the solution under question has no symmetry whatsoever. An overlooked property is thereby emphasized: the general relativity of 1915 is more appropriately considered to be a theory whose intrinsic "relativity content" is not given a priori once and for all, like it happens in the special relativity of 1905. Its true content can be ascertained only in a case by case way, after its solutions are found, and the elements of the Killing group are determined by solving the Killing equations (2.6). Different solutions, i.e. different manifolds can and do exhibit a different relativity postulate, possibly a vanishing relativity content. Moreover, different submanifolds of a given manifold can and sometimes do exhibit Killing groups with a different relativity postulate in which, according to the well known results by Noether [7], different conservation laws can and do prevail.

5. The peculiar Killing group of the static solutions of general relativity

A physically quite relevant example of solutions to the field equations of general relativity with a nontrivial Killing group is given by the so-called static solutions. The notion of staticness as an intrinsic feature independent from coordinates was clearly in the mind of a mathematician like Levi-Civita when he introduced and discussed at length static solutions given in symmetry-adapted static coordinates, shown in his ground-breaking work [14] on "Einsteinian statics" and in the series of eight Notes [15]-[16], all entitled "Einsteinian ds^2 in Newtonian fields". The direct, intrinsic definition of staticness through the perusal of the "static Killing group" only appeared much later in the chapter published in 1964 by Ehlers and Kundt [17]. We shall follow the intrinsic definition of staticness, more precisely, the definition of *static vacuum fields* through their Killing vectors, since it is the only way to enlighten a property of uniqueness, that was given explicitly in [17] for the first time, and is crucial for grasping the quite novel physical property of staticness as it occurs in general relativity. It is exhibited by the exact vacuum solutions, but its peculiarity is usually paid scarce attention in the literature. In [17], after having proved

Theorem 2-3.1: "A space-time is static if and only if it admits a group G_1 of isometries whose trajectories form a time-like, normal congruence."

on page 65 Ehlers and Kundt reach theorem 2-3.2 and the crucial

Corollary 1. "In a static space-time there exist precisely one static congruence, and precisely one G_1 with time-like, hypersurface-orthogonal trajectories provided that either the conform tensor is non-degenerate or the time-like eigenvector of the Ricci tensor is simple."

From these results it transpires that the notion of staticness in general relativity has a deep meaning that directly stems from the peculiar structure of the Killing group decided by solutions to the field equations, and finds no

counterpart in the different Killing structure of the infinitesimal Poincaré group set a priori in special relativity.

In the latter theory, through a given event, an infinite number of distinct timelike Killing vectors can be drawn. To any such vector, through a given event one and just one spatial hypersurface can be found, that is orthogonal to the chosen timelike Killing vector. This means that the foliation of spacetime in space and time can be performed in infinite ways in special relativity, i.e. it has no intrinsic character [18]. Through an infinitesimal Poincaré transformation, or through a sequence of these transformations, any one of the distinct timelike Killing vectors can be brought to rest in the coordinate and in the reference frame sense. This occurs in keeping with the very notion of relativity of motion that prevails in special relativity: no absolute rest can be defined in an intrinsic way in the Minkowski metric manifold.

According to Corollary 1, the opposite is true for the ample class of vacuum solutions of general relativity that are named static after Ehlers and Kundt. Through a given event of such a solution, provided that either the conform tensor is non-degenerate or the time-like eigenvector of the Ricci tensor is simple, a unique timelike Killing vector exists, that is hypersurface-orthogonal too. From it, one and just one static congruence is defined through the given event. When this is the case, the foliation of spacetime in space and time, that is frame-dependent in special relativity [18], is instead uniquely given at each event. This foliation is an absolute one, an absolute physical property intrinsic to the manifold. When gravitation is present, by measuring the metric in principle one can decide whether a test body is intrinsically at rest or not with respect to the manifold under question.

Solutions to the vacuum field equations of general relativity that are static in the sense of Ehlers and Kundt do exist. Solutions belonging to the class found by Weyl [19] and by Levi-Civita [16] have been proved [17] to be static in that sense. It is mandatory to interpret physically the static solutions of the field equations of general relativity by paying due attention to the peculiar uniqueness property exhibited by their Killing group.

6. Choosing the manifold of Schwarzschild's solution

In general relativity, it is obvious that the manifold to be associated to a given solution cannot be chosen by deciding a priori the ranges of the coordinates in certain charts. Since coordinates are mere labels, the choice of the manifold must always rely on intrinsic physical arguments that need to be developed a posteriori, once a solution to the field equations is found. To this end the nontrivial Killing structure of the solution needs to be investigated. Its outcome may be crucial for the very choice of the manifold. As a corollary to the results of the previous sections, the example of the intrinsically motivated choice of the manifold that necessarily applies to the

Schwarzschild solution [20], [21], when the properties of its Killing group are kept into account, is given here.

In the symmetry-adapted coordinates $x^1 = r$, $x^2 = \vartheta$, $x^3 = \varphi$, $x^4 = t$ chosen by Hilbert [21], the interval of Schwarzschild's solution reads

$$(6.1) \quad \mathrm{d}s^2 = \left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 - \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 - r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\,\mathrm{d}\varphi^2),$$

where $m > 0$ agrees with the mass of a body in the Newtonian limit. Due to the spherical Killing symmetry of the solution, we shall set $-\pi/2 \le \vartheta \le \pi/2$ and $0 < \varphi \le 2\pi$. When $r > 2m$, the solution is static in the sense of Ehlers and Kundt. Like it occurs with the Weyl-Levi Civita solutions, at any event a unique hypersurface-orthogonal wordline of absolute rest is drawn, defined in the chosen coordinates through constant values of r, ϑ and φ. Let us consider a test body on a certain worldline of the manifold, whose four-velocity is $u^i \equiv \frac{\mathrm{d}x^i}{\mathrm{d}s}$; its acceleration four-vector, i.e. the first curvature of its worldline [22], is defined as

$$(6.2) \qquad\qquad a^i \equiv \frac{\mathrm{D}u^i}{\mathrm{d}s} \equiv \frac{\mathrm{d}u^i}{\mathrm{d}s} + \Gamma^i_{kl}u^k u^l,$$

where $\mathrm{D}/\mathrm{d}s$ indicates the absolute derivative. From it, one builds the scalar quantity

$$(6.3) \qquad\qquad \alpha = (-a_i a^i)^{1/2}.$$

When the test body lies on a worldline of absolute rest, the norm of its four-acceleration, written in the chosen coordinates, reads

$$(6.4) \qquad\qquad \alpha = \left[\frac{m^2}{r^3(r - 2m)}\right]^{1/2}.$$

Through a given event, α is uniquely defined by the Killing structure of the manifold [17]. Therefore it is clear that this quantity, besides being an invariant, is intrinsic to the manifold. In fact, due to the uniqueness of the hypersurface-orthogonal time Killing vector, in the definition of α no arbitrary choices based on elements foreign to the structure of the manifold, like the arbitrary choice of a certain worldline, have been invoked. The acceleration is measurable through the power of an engine trying to keep a rocket stationary. Nevertheless, one should not call this a singularity, as M.McCallum noted in [28], because this notion is reserved for curvature properties, even if Schwarzschild debated this way [20].

7. Further remarks on Schwarzschild's manifold

One should remember that, when the group-theoretical argument recalled in the previous Section is overlooked and, like it happened with the choice of the manifold done by Hilbert [21] in his reinterpretation of Schwarzschild's original solution, the range of the radial coordinate r is assumed to be $0 < r < +\infty$, a pathology soon appears. It originates from the difference in the

Killing groups prevailing for $r > 2m$ and for $r < 2m$ respectively, and it can be avoided only if the changes of topology of Hilbert's manifold produced by the maximal extensions of Synge [23], Kruskal [24] and Szekeres [25] are introduced. This pathology is usually given scarce relevance in the literature, although e.g. Rindler did not forget to mention it in his book [26]. However, in order to appreciate its full meaning, one has rather resorting to Synge, were a detailed discussion of the issue of the time arrow in the Schwarzschild solution can be found [23]. In keeping with Synge, a manifold meant to be a model of physical reality must fulfill two postulates. One of them is the *postulate of order*, according to which the parameter of proper time along a timelike geodesic must always either decrease or increase; the sense along which it is assumed to increase defines the sense of the travel from past to future, namely the time arrow. Since the geodesic equation (3.1) is quadratic in the line element, fixing the time arrow of the individual geodesic is a matter of choice. The second postulate deals with our ideas of causation, and establishes a relation between the time arrows of neighbouring geodesics. Synge calls it the *non-circuital postulate*. It asserts that *there cannot exist in space-time a closed loop of time-like geodesics around which we may travel always following the sense of the time-arrow.*

Synge was the first to show in detail [23] that the time arrow can be drawn in keeping with the aforementioned postulates in the maximally extended manifold that he obtained from the Hilbert manifold with his singular coordinate transformation; the same property obviously holds in the Kruskal-Szekeres manifold too. Does it hold also in the Hilbert manifold? A glance to Figure (1a) is sufficient to answer the question in the negative. The arrow of time can be drawn in keeping with Synge's two postulates of order and of non-circuitality in the submanifold with $2m < r < +\infty$, i.e. in Schwarzschild's original manifold, and separately in the inner submanifold with $0 < r < 2m$. A consistent drawing of the arrow of time, in keeping with both postulates, is however impossible in Hilbert's manifold as a whole. This is an intrinsic flaw of the latter manifold, originating from the abrupt change in the Killing group structure that occurs at the crossing of the surface $r = 2m$, where the unique, hypersurface-orthogonal, timelike Killing vector suddenly becomes spacelike. It has nothing to do either with the fact that in Hilbert's chart the metric is not defined at $r = 2m$, or with the fact that in it the timelike geodesics appear to cross the Schwarzschild surface at the coordinate time $t = \pm\infty$; it is a flaw that cannot be remedied by *any* coordinate transformation, however singular at $r = 2m$, but one-to-one elsewhere.

The only known ways to overcome this flaw of Hilbert's manifold are either by eliminating the inner region, thereby reinstating the choice of the original manifold [20], deliberately made by Schwarzschild as a model for the gravitational field of a material particle, and later confirmed by the study of the Killing group structure [27], or by completely renouncing the one-to-one

injunction on the coordinate transformations once set, on physical grounds, by Einstein [11] and by Hilbert [3].

The second alternative is the one chosen by Synge and his followers: in fact, not only Schwarzschild's original manifold, but also Kruskal's manifold avoids the flaw of the arrow of time present in Hilbert's manifold. Moreover, it appears to preserve its inner region, for which $0 < r < 2m$. However, it does so by a coordinate transformation that duplicates the original manifold and alters its topology, in a way that is best explained, rather than by looking at the equations for the transformations, through a straighforward cut-and-paste procedure applied to two Hilbert manifolds.
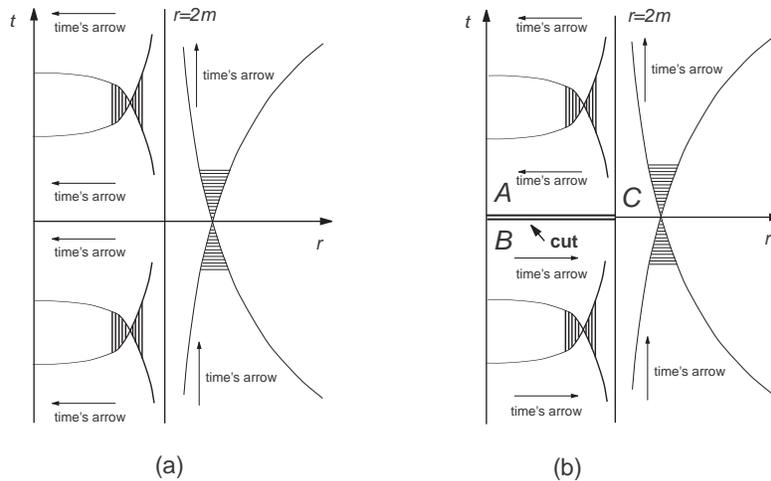

FIGURE 1. (a): Drawing of Hilbert's manifold in the r, t plane. Light cones are drawn both for $r < 2m$ and for $r > 2m$. Time arrows are drawn in agreement with the non-circuital postulate. Then the postulate of order happens to be violated. (b): Hilbert's manifold is cut along AC. The topologically different manifold obtained in this way allows for a drawing of the time arrow in keeping with both Synge's postulates.

This procedure can be made in infinite ways, all entailing the same change of topology. One of them is accounted for in the sequence drawn in Figures (1b), (2a) and (2b) respectively. In Figure (1b) the inner region of the Hilbert manifold of Figure (1a) is cut along the line AC of the r, t diagram. The resulting manifold is topologically inequivalent to the Hilbert manifold. The topological alteration already allows to draw the arrows of time in keeping with both Synge's postulates but, due to the existence of the border ACB, the new manifold is evidently unphysical. However, if one takes two manifolds identical to the one of Figure (1b), juxtaposes them as it is shown in Figure (2a), and eventually sews together the borders ACB and $A'C'B'$ like in Figure (2b), one obtains a manifold equal to Kruskal's manifold,
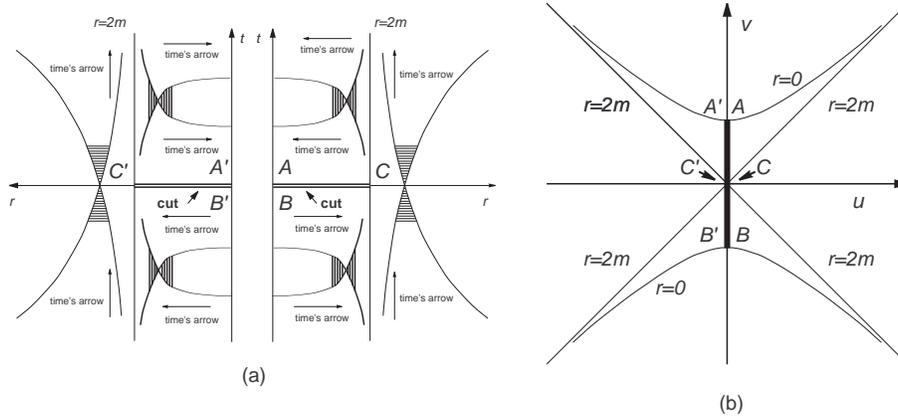
FIGURE 2. (a): Two manifolds, equal to the Hilbert manifold cut
along ACB and $A'C'B'$ in Figure (1b) are juxtaposed for suturing;
(b): sewing together the edges ACB and $A'C'B'$ produces Kruskal's
manifold, that fullfils both Synge's postulates.

and ascertains that the arrows of time inherited from the two component
manifolds with the cut still obey both Synge's postulates.

Only the topological alteration, drawn in the Figures, that leads from
the Hilbert manifold to the Kruskal manifold remedies the flaw of the time
arrow due to the coexistence, in the Hilbert manifold, of two submanifolds
with a different Killing group structure.

8. CONCLUSION

The relation between relativity and invariance was clarified long ago by
the work of Felix Klein [4],[5] and his result constitutes a paradigm that
goes beyond the limits of special relativity theory. Einstein's idea, that one
should introduce general transformations between curvilinear coordinates
was fundamental from a mathematical standpoint, since it allowed one to
avail of the powerful methods of the absolute differential calculus of Ricci
and Levi-Civita [9]. However, the physical idea by Einstein [11], that cer-
tain curvilinear coordinates should be availed of to account for non-inertial
reference frames, as required by the early form of the equivalence principle,
did not resist the criticism by Kretschmann [6]. General curvilinear coordi-
nates are very useful mathematical tools, but they are physically vacuous.
The group of general covariance is physically vacuous too. By following the
ideas of Klein and Kretschmann, one shall trace, in general relativity like in
special relativity, the physically meaningful group of invariance. The Killing
group of infinitesimal "Mitschleppen" of the metric tensor is such a group.
In a general solution of the field equations of 1915, the Killing group is triv-
ial, and general relativity then behaves like an absolute theory. However,

solutions with Killing groups that are intermediate between the trivial one pertaining to an absolute solution and the infinitesimal Poincaré group do exist. Their scrutiny is fundamental for assessing the very structure of the manifolds from a physical standpoint. When this scrutiny is applied to the Schwarzschild solution, it turns out that only the manifold originally chosen by Schwarzschild [20] survives. Other manifolds, like the ones chosen by Hilbert [21], and later by Synge [23], by Kruskal [24] and by Szekeres [25] with their maximal extensions cannot survive the scrutiny, because they contain a local, invariant, intrinsic quantity that diverges in their interior.

References

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