GRAVITATIONAL FIELDS WITH NULL POINTS OF THE DETERMINANT OF $g_{\mu\nu}$. II

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Abstract. We shall study the geometrical properties of the static Einstein spaces with null points of the determinant g introduced in the first part. It results in particular that these Einstein spaces are incomplete. It will be further shown that nothing essentially changes, with respect to the results of the first part if, instead of vacuum gravitational fields, one considers gravitational fields coupled with material fields. - Physical consequences will be discussed, that result from the global structure of the incomplete V_4. The static incomplete V_4 with null points of g prove to be field theoretical models of pointlike particles in the sense of Einstein.

1. The signature of space as field function

In a previous work of H. Treder[1], the signature s of the physical space-time manifold V_4 is considered as a field function, that can have a different character from place to place. Since the signature s is given by the difference between the number of negative and of positive eigenvalues of the metric tensor $g_{\mu\nu}$, s is necessarily a discontinuous function. In I it has been proposed in particular to assume as interior of a particle a spatially limited region (cylinder in V_4), in which V_4 is endowed with a negative definite metric. The index of inertia then reads $(-1, -1, -1, -1)$ and the signature turns out to be $s = -4$. Outside a particle it holds the usual signature $s = -2$ of Lorentz and Minkowski, with the index of inertia $(-1, -1, -1, +1)$.

Inside the particle the determinant g of $g_{\mu\nu}$ is therefore positive, and outside it is $g < 0$. Due to the requirement of continuity for $g_{\mu\nu}$, it exists a hypersurface S that divides the internal region from the external one, and on which g vanishes. On S the index of inertia reads $(-1, -1, -1, 0)$ and we have a semidefinite metric with $s = -3$. - S represents the hystory of the surface of the particle.

We give to S the representation $z(x^\nu) = 0$; we denote the interior of the particle with $z < 0$, and the exterior region with $z > 0$. Then the

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[1]H. Treder, Ann. Physik (7), **9**, 283 (1962). Cited in the following as I; see also H. Treder in Proceedings of the Jablonna Conference on General Relativity, Warsaw (in print).

discontinuous field function s has the following behaviour:

$$s = -2 \text{ for } z > 0, \quad s = -3 \text{ for } z = 0, \quad s = -4 \text{ for } z < 0. \tag{1}$$

It has been shown in I that the null points of g on S do not constitute a singularity of Einstein's vacuum equations. On the contrary, these equations are satisfied in the sense of a limit also at null points of g of finite (n-th) order. If conditions are imposed on the form of S, from Einstein's equations follow conditions for the possible ways of vanishing of g.

If these conditions give that it must be $n = 2m$ (m = natural number), also for $z < 0$ it is $g < 0$, and we have a V_4 that, apart from the hypersurface S, where g vanishes, is endowed with the Minkowski signature $s = -2$. As remarked in I, in this case we shall pose $z \equiv \varrho$, we shall interpret ϱ as radial coordinate and $\varrho = 0$ as the origin of a system of radial coordinates, so that there is no region with $z < 0$.

We thus obtain a field theoretical model of a point particle in which, according to Einstein's proposal[2], the worldline of the particle is represented by the line $g = 0$.

Both the case $g \begin{smallmatrix} \geq \\ < \end{smallmatrix} 0$ and the case $g \leq 0$ will provide the geometry of particle models in the sense of Einstein's program[3]: regions with strong gravitational fields are substituted either for discrete material distributions, or for the delta-like field singularities of equivalent meaning, hence in general for regions in which the field equations are not satisfied. - Since, according to Sylvester's theorem of inertia, the signature s of V_4 cannot be changed by any regular, real transformation of coordinates, in our hypothesis the extreme strength of the gravitational field is an invariant property with respect to all the regular transformations. - In the region with $g \geq 0$ the gravitational field deviates totally from the Minkowski space, since not even in the infinitesimal neighbourhood of a world point P it can be brought to the form $g_{\mu\nu} = \eta_{\mu\nu}$ + small terms, where

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & -1 & \\ & & & +1 \end{pmatrix}$$

is the Minkowski tensor.

In a V_4 with the usual Minkowskian signature Einstein's idea, namely substituting strong gravitational fields for the particles, strikes against a peculiar difficulty, that cannot be overcome at least for point particles: by constructing Fermi coordinates one can get $g_{\mu\nu} = \eta_{\mu\nu}$ along the woldline $x^\nu = x^\nu(\tau)$ of the "particle", thereby wholly eliminating the gravitational

[2] A. Einstein and N. Rosen, Physic. Rev. **48**, 73 (1935); A. Einstein, Jour. Franklin Inst. **221**, 313 (1936).

[3] see for instance A. Einstein, Philosopher-Scientist, New York 1951, p. 81; A. Einstein, The Meaning of Relativity, Princeton 1955, pp. 163-166.

field for $x^\nu = x^\nu(\tau)$. - This difficulty does not exist in a model for which the worldline of a point particle is given by $g = \varrho^{2m}$=0. When $g = 0$, $g_{\mu\nu} = \eta_{\mu\nu}$ ($x^\nu = x^\nu(\tau)$) is not reachable through any regular transformation.

In I static vacuum fields with null points of g have been studied. We have there dealt with the following geometrical problem:

The Einsteinian V_4 possesses a Killing vector ξ^μ, that will define a family of subspaces V_3. ξ^μ is timelike in the region of V_4 where the Minkowski signature holds. The subspaces V_3 are the usual three-dimensional spaces (*i.e.*, for $g < 0$, V_4 is static in the sense of Levi-Civita). It exists a family S' of hypersurfaces (timelike for $g < 0$) on which ξ^μ lies. These surfaces S' are closed in two dimensions, hence they are spatiotemporal cylinders. - At a surface S of the family S', however, ξ^μ becomes a null vector, and inside S it will become spacelike.

Since the subspaces V_3 will have a negative-definite metric, on S there are no timelike directions, and only one null direction, which is the direction of the Killing vector. The signature of S is $s = -3$ with the index $(-1, -1, -1, 0)$. Inside S there are only spacelike directions. Under these hypotheses, in keeping with I, the metric in proximity of S can be specialised as:

$$g_{\mu\nu} = g_{\mu\nu}(x^i) \quad i = 1, 2, 3, \tag{2}$$

$$g_{i4} = 0, \quad |g_{ik}| < 0. \tag{3}$$

We pose further

$$z(x^\nu) = f(x^i) = x^1. \tag{4}$$

Then, according to the hypotheses

$$g = |g_{ik}| g_{44} = |g_{ik}|_{x^1=0} \left(\alpha_n(x^2, x^3)(x^1)^n + \cdots\right), \tag{5}$$

i.e. due to (3):

$$g_{44} = \alpha_n(x^2, x^3)(x^1)^n + \cdots. \tag{6}$$

Furthermore, we can also obtain

$$g_{1a} = 0, \quad a = 2, 3. \tag{7}$$

Since we require that g and hence g_{44} have everywhere on S a zero of order n, we can also pose in particular:

$$g_{44} = (x^1)^n. \tag{8}$$

The general form (6) is a physically inessential complication. If α_n in (6) vanishes everywhere, instead of (8) one poses simply

$$g_{44} = (x^1)^{n'}, \quad n' = n + 1. \tag{9}$$

If instead $\alpha_n(x^2, x^3) = 0$ has discrete solutions $x^a = c^a$, the subsequent calculations (§§2 and 3) hold everywhere, except for the corresponding points, for which the final result: "The static V_4 is incomplete and describes pointlike particles (with constant x^a)" does not apply. - In I it has been shown that from the Einstein vacuum equations it follows $g_{44} = (x^1)^2 = \varrho^2$.

In §2 we shall extend the calculation of I to the case when, besides the static gravitational field $g_{\mu\nu}(x^i)$, an electromagnetic field $F_{\mu\nu}$, spinor fields and scalar meson fields exist too. Furthermore we shall assume that these fields are mutually coupled in the usual manner. We require that all the covariant field quantities shall be limited. On the contrary, the contravariant ones can be unlimited for $x^1 \to 0$, when they result from the covariant ones through raising with g^{44}. We shall then see that the Einstein equations

$$E_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa T_{\mu\nu} \tag{10}$$

give $n = 2$ also in this case, hence the particles are pointlike and one must pose $x^1 \equiv \varrho$.

At first sight the equations (10) have no meaning for $g = 0$. In fact the Ricci tensor $R_{\mu\nu}$ contains terms $\sim g^{-2}$ and $R = g^{\sigma\tau} R_{\sigma\tau}$ terms $\sim g^{-3}$, hence the Einstein tensor $E_{\mu\nu}$ can diverge like g^{-3} for $x^1 \to 0$. Nevertheless (like in I) the field equations (10) hold on S in the sense of a limit. - If we assume (as it happens through the choice of the field sources), that $T_{\mu\nu}$ diverges at most like g^{-3}, we can substitute for (10) an equation of the Einstein-Rosen type:

$$g^3 \left(E_{\mu\nu} + \kappa T_{\mu\nu} \right) = 0. \tag{11}$$

The density $g^3 E_{\mu\nu}$ is defined also for $g \to 0$. - If $T_{\mu\nu}$ would diverge like g^{-p} - with $p > 3$ - instead of (11) we should simply write

$$g^p \left(E_{\mu\nu} + \kappa T_{\mu\nu} \right) = 0. \tag{12}$$

Equation (11) holds now both on S and in a finite layer on both sides of S, hence in a region $-\varepsilon \leq x^1 \leq +\varepsilon$. With the assumption $g \sim (x^1)^n$, that is always possible when g vanishes with a finite order, according to l'Hôpital's rule we can form the limit

$$\lim_{x^1 \to 0} \left(E_{\mu\nu} + \kappa T_{\mu\nu} \right) = \lim_{x^1 \to 0} \left(\frac{\frac{\partial^{3n}}{\partial (x^1)^{3n}} \left[g^3 \left(E_{\mu\nu} + \kappa T_{\mu\nu} \right) \right]}{\frac{\partial^{3n}}{\partial (x^1)^{3n}} g^3} \right). \tag{13}$$

Hence (11) holds also in the position $x^1 = 0$, if it holds for $x^1 > 0$ and for $x^1 < 0$.

Since the inner region $g \geq 0$ does not contain any timelike direction, no particle can enter the inner region from the outer one. Inside the inner region all the reactions happen instantaneously without retardation. Hence the inner region $g \geq 0$ is both a "black sphere" and an ideal "rigid body" of the classical mechanics.

Geometrically it is relevant that, due to the nonexistence of a timelike direction, no timelike geodesics can enter the region $g \geq 0$. Timelike geodesics, that come arbitrarily close to the surface (or curve) $x^1 = 0$, must approach it asymptotically. - In §3 we shall show that this has for consequence that the V_4 determined in I and in §2 are incomplete manifolds, since in them there are timelike geodesics with a finite branch.

If such a geodesic is the worldline of a test particle, this fact means, for an observer at infinity, absorption or irregular diffusion of the test particle (see §3).

The results of I and of §§2 and 3 allow for some general remarks over the meaning of incomplete V_4 for Einstein's particle problem, that will be discussed in §4.

2. The general static field

We shall show now that, for static gravitational fields, for which the matter tensor $T_{\mu\nu}$ corresponds to the known physical fields, the determinant of $g_{\mu\nu}$ must have a null point of second order on the hypersurface S. For the field intensity $F_{\mu\nu}$ of Maxwell's field and for the spinor fields ψ we admit as usual single continuous differentiability, and for the meson fields φ twofold continuous differentiability.

One gets simplifications for $g_{\mu\nu}$ from the existence of the one-dimensional symmetry group with hypersurface orthogonal Killing vector. We can avail here of the results of I. In I (§3, equations 22, 28, 29, 30, 31 and §1, equation 8) $g_{\mu\nu}$ and $g^{\mu\nu}$ can be written in the following form:

$$g_{i4} = 0,\ g^{i4} = 0,\ g_{44} = \frac{1}{g_{44}},\ g^{11} = \frac{1}{g_{11}} \neq 0, \tag{14}$$

$$g_{ik} = \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & g_{23} \\ 0 & g_{23} & g_{33} \end{pmatrix},\quad g^{ik} = \begin{pmatrix} g^{11} & 0 & 0 \\ 0 & g^{22} & g^{23} \\ 0 & g^{23} & g^{33} \end{pmatrix}, \tag{15}$$

$$g_{44} = (x^1)^n. \tag{16}$$

We shall further assume that $|g_{ik}| \neq 0$. Then the only component of the gravitational potential that has a pole on S is g^{44}.

With the abbreviations $g_{44} = V^2$ and $|g_{ik}| = -\gamma^2$ we get for the Ricci tensor:

$$\begin{aligned} R_{44} &= \frac{V}{\gamma}\left(g^{ik}\gamma V_{,i}\right)_{,k} = 0,\ R_{i4} = 0, \\ R_{kl} &= P_{kl} + \frac{1}{V}\left(V_{,kl} - \Gamma^i_{kl} V_{,i}\right). \end{aligned} \tag{17}$$

Here P_{ik} denotes the three-dimensional Ricci tensor, formed with g_{ik}. - We shall deal in the following with Einstein's equations in presence of matter:

$$E_{\mu\nu} = -\kappa T_{\mu\nu}.$$

We first calculate Einstein's tensor by starting from (17). We get from (17), with the form of $g_{\mu\nu}$ given by (14), (15), (16):

$$R = g^{kl}R_{kl} + g^{44}R_{44} = P + \frac{2g^{mn}}{V}\left(V_{,mn} - \Gamma^{i}_{mn}V_{,i}\right), \tag{18}$$

where $P = g^{kl}P_{kl}$. We avail here of the relation

$$\frac{1}{\gamma}\left(g^{ik}\gamma\right)_{,k} = -g^{kl}\Gamma^{i}_{kl}.$$

From (17) and (18) it results for $E_{\mu\nu}$:

$$E_{44} = -\frac{1}{2}PV^2,\ E_{k4} = 0, \tag{19}$$

$$E_{kl} = P_{kl} - \frac{1}{2}g_{kl}P + \frac{1}{V}\left(V_{,kl} - \Gamma^{i}_{kl}V_{,i} - g^{mn}g_{kl}V_{,mn} + g_{kl}g^{mn}\Gamma^{i}_{mn}V_{,i}\right). \tag{20}$$

We shall assume that $T_{\mu\nu}$, the matter tensor of the known physical fields (Dirac's fields, scalar meson fields, Maxwell field), has the usual couplings. $T_{\mu\nu}$ is derivable from a Lagrange function that can be written as follows:

$$\begin{aligned} L = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + g^{\mu\nu}\sum_j\left[D^*_{(\mu}\varphi^*_j D_{\nu)}\varphi_j + \frac{1}{2}\nabla_{(\mu}\varphi^0_j\nabla_{\nu)}\varphi^0_j\right] \\ +\frac{i}{2}g^{\mu\nu}\sum_N\left[\overline{\psi}_N\gamma_{(\nu}\nabla_{\mu)}\psi_N - \nabla_{(\mu}\overline{\psi}_N\gamma_{\nu)}\psi_N\right] \\ +\frac{i}{2}g^{\mu\nu}\sum_P\left[\overline{\psi}_P\gamma_{(\nu}D_{\mu)}\psi_P - D^*_{(\mu}\overline{\psi}_P\gamma_{\nu)}\psi_P\right] \\ -\sum_j\varphi^*_j\varphi_j m^2_j - \sum_j\frac{{m^0}^2_j}{2}{\varphi^0}^2_j \\ -\sum_P\mu_P\overline{\psi}_P\psi_P - \sum_N\mu_N\overline{\psi}_N\psi_N + W. \end{aligned} \tag{21}$$

$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ is the electromagnetic field. ∇_μ is the operator of covariant differentiation, and $D_\mu = \nabla_\mu - ieA_\mu$ the operator of gauge invariant differentiation (one shall care about the fact that ∇_μ takes a different meaning according to the kind of field quantity over which ∇_μ operates). The γ_μ are Dirac's spin matrices, with $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$. The φ_j (resp. φ^0_j) mean the different scalar fields, respectively charged and neutral. m_j, m^0_j, μ_N and μ_P are the masses of the corresponding particles. In (21) we have further assumed that the scalar fields and the Dirac fields are mutually coupled in the usual way. These couplings (Yukawa coupling of the form $\overline{\psi}\gamma_5\tau_\varrho\psi\varphi_\varrho$ and universal Fermi coupling of the form $\overline{\psi}(1+\gamma_5)\psi_B\overline{\psi}_C(1-\gamma_5)\psi_D$) are indicated in (21) with the term W, independent of $g_{\mu\nu}$.

The variation of $\mathcal{L} = \sqrt{-g}L$ with respect to $g^{\mu\nu}$ yields, with an elementary calculation:

$$\sqrt{-g}T_{\mu\nu} = \frac{2\delta\mathcal{L}}{\delta g_{\mu\nu}} = \sqrt{-g}\left(2G_{\mu\nu} - F_{\mu}{}^{\alpha}F_{\nu\alpha} - g_{\mu\nu}L\right). \tag{22}$$

Here $G_{\mu\nu}$ is given by[4]

$$\begin{aligned} G_{\mu\nu} &= \sum_j \left[D^*_{(\mu}\varphi^*_j D_{\nu)}\varphi_j + \frac{1}{2}\nabla_{(\mu}\varphi^0_j\nabla_{\nu)}\varphi^0_j\right] \\ &+\frac{i}{2}\sum_N \left[\overline{\psi}_N\gamma_{(\nu}\nabla_{\mu)}\psi_N - \nabla_{(\mu}\overline{\psi}_N\gamma_{\nu)}\psi_N\right] \\ &+\frac{i}{2}\sum_P \left[\overline{\psi}_P\gamma_{(\nu}D_{\mu)}\psi_P - D^*_{(\mu}\overline{\psi}_P\gamma_{\nu)}\psi_P\right]. \end{aligned} \tag{23}$$

With the abbreviation

$$G_0 = -\sum_j \left(\varphi^*_j\varphi_j m^2_j + \frac{m^{0\,2}_j}{2}\varphi^{0\,2}_j\right) - \sum_P \mu_P\overline{\psi}_P\psi_P - \sum_N \mu_N\overline{\psi}_N\psi_N + W,$$

from (19), (20), (21), (22) and (23), by taking into account (14), (15), one gets the following field equations for the gravitational field:

$$\begin{aligned} E^4_4 &= -\frac{1}{2}P \\ &= -\kappa\left(2g^{44}G_{44} - F_{4s}F_{4r}g^{sr}g^{44} - \left[G_0 + g^{\mu\nu}G_{\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right]\right), \end{aligned} \tag{24}$$

$$\begin{aligned} E^l_k = g^{rl}E_{kr} &= g^{rl}P_{kr} - \frac{1}{2}\delta^l_k P + g^{rl}\frac{1}{V}\left(V_{,kr} - \Gamma^i_{kr}V_{,i}\right) \\ &-\delta^l_k\frac{1}{V}g^{mn}\left(V_{,mn} - \Gamma^i_{mn}V_{,i}\right) = -\kappa T^l_k, \end{aligned} \tag{25}$$

with

$$\begin{aligned} T^l_k &= 2g^{rl}G_{rk} - g^{rl}F_{r\sigma}F_{k\alpha}g^{\alpha\sigma} \\ &-\delta^l_k\left(G_0 + g^{\mu\nu}G_{\mu\nu} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}\right), \end{aligned} \tag{26}$$

and

$$E^4_k = E^k_4 = -\kappa T^4_k = -\kappa T^k_4 = 0. \tag{27}$$

We shall now compare the singular terms at $x^1 = 0$. We shall write always only the divergent terms of (24), (25). We avail of the fact that G_0, A_μ, $F_{\mu\nu}$ and g^{kl} cannot contain any singularity, since the covariant fields are by hypothesis limited for $x^1 = 0$ and $g^{44} = 1/V^2$ is, according to (14), (15), (16), the only singular term.

[4]Equations (22), (23) are very easily verified with the help of the equivalence principle, by starting from the form of $T_{\mu\nu}$ that holds in special relativity, and then by writing it in covariant form.

By covariant differentiation of the spinors ψ, $G_{\mu\nu}$ can instead contain further singular terms. One has

$$\nabla_\nu \psi \equiv \psi_{,\nu} - \Gamma_\nu \psi. \tag{28}$$

The Γ_μ are the coefficients for the parallel transportation of spinors, for which one has:

$$\gamma_{\mu,\nu} - \Gamma^{\varrho}_{\mu\nu}\gamma_\varrho + [\gamma_\mu, \Gamma_\nu] = 0. \tag{29}$$

We shall choose the γ_μ in a particular way, so that they can be obtained[5] from the Dirac matrices γ^0_μ of special relativity according to

$$\gamma_\mu = a^{\nu}_{\mu}\gamma^0_\nu. \tag{30}$$

From (14) and from $g_{\mu\nu,4} = 0$ one gets $\Gamma_{4,il} = \Gamma_{i,4l} = 0$. Furthermore we have from (14), (15) and from (30)

$$\gamma_4 = \sqrt{g_{44}}\gamma^0_4, \tag{31}$$

by keeping into account that $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$. From (29) with $\nu = l$ one gets:

$$0 = [\gamma_k, \Gamma_l] + \gamma_{k,l} - \Gamma^i_{kl}\gamma_i, \tag{32}$$

$$0 = [\gamma_4, \Gamma_l] + \gamma_{4,l} - \frac{1}{2} g_{44,l}\gamma^4 = [\gamma_4, \Gamma_l]. \tag{33}$$

From this it follows immediately that Γ_l must be limited up to a vector that commutes with all the γ_μ, because Γ_l is defined by quantities that are limited for $x^1 = 0$ (we set equal to zero the undetermined vector in Γ_ν, since we deal separately with the electromagnetic potential A_μ, according to the distinction between charged and neutral Dirac fields.). From (29) one instead gets for Γ_4:

$$0 = [\gamma_k, \Gamma_4] - \frac{1}{2} g^{44} g_{44,k}\gamma_4, \tag{34}$$

$$0 = [\gamma_4, \Gamma_4] - \Gamma^i_{44}\gamma_i. \tag{35}$$

One sees that Γ_4 must behave like $(x^1)^{\left(\frac{n}{2}-1\right)}$ for $x^1 \to 0$; hence, for $n < 2$ it can diverge when $x^1 = 0$.

Therefore, while the terms G_{kl} are limited, due to its dependence on Γ_4 we must keep into account G_{44} in the comparison of the terms that are singular for $x^1 = 0$. From (24), (25) it follows:

$$E^4_4 = -\frac{1}{2}P = -\kappa T^4_4 \sim -\kappa\left(-\frac{1}{2}F_{4k}F_{4r}g^{kr}g^{44} + g^{44}G_{44}\right) \sim -\kappa L, \tag{36}$$

[5] see V. Bargmann, Berliner Berichte 1932, p. 346.

(as one can see, we must consider G_{44} also due to the term $G_4^4 = g^{44}G_{44}$.)

$$E_l^k \sim g^{1l}\frac{1}{V}\left(V_{,k1} - \Gamma^i_{k1}V_{,i}\right) - \delta^l_k g^{11}\frac{1}{V}\left(V_{,11} - \Gamma^i_{11}V_{,i}\right)$$

$$+g^{al}\frac{1}{V}\left(V_{,ka} - \Gamma^i_{ka}V_{,i}\right) - \delta^l_k g^{ab}\frac{1}{V}\left(V_{,ab} - \Gamma^i_{ab}V_{,i}\right) \sim -\kappa T^l_k, \tag{37}$$

where

$$T^l_k \sim -g^{rl}F_{k4}F_{r4}g^{44} - \delta^l_k L.$$

Since E_4^4 is evidently limited, due to (36) the expression $\delta^l_k L$ in T^l_k is limited and one gets, as divergent term of T^l_k:

$$T^l_k \sim g^{rl}F_{k4}F_{r4}g^{44} \sim g^{rl}F_{k4}F_{r4}(x^1)^{(-n)}. \tag{38}$$

Now, for $x^1 = 0$, $g^{44} = 1/V^2$ has a pole of order n. Hence $V_{,1}/V$ contains a pole of first order and $V_{,11}/V$ contains a pole of second order. From (37) one sees that, for $l \neq k$, in E^l_k appear only first derivatives of V with respect to x^1, and in this case E^l_k has at most poles of first order. Also E^1_1 contains at most poles of first order, because the terms with $V_{,11}/V$ cancel exactly. E^2_2 and E^3_3 have instead poles of second order, with the form:

$$E^2_2 \sim E^3_3 \sim -\frac{V_{,11}}{V}g^{11} \sim -\frac{n}{2}\left(\frac{n}{2} - 1\right)(x^1)^{-2}g^{11}. \tag{39}$$

Thanks to (37), (38), the comparison between E^1_1 and T^1_1 shows that F_{14} must have a zero of order $\varrho \geq \frac{n-1}{2}$, because E^1_1 for $x^1 = 0$ has at most a pole of first order. Given the form of E^3_2 and of E^2_3, also T^3_2 and T^2_3 can have at most poles of the first order. Due to (38) one therefore obtains:

$$F_{34}\left(g^{22}F_{24} + g^{23}F_{34}\right) = 0 \quad \text{of order } \varrho_1 \geq n - 1, \tag{40}$$

$$F_{24}\left(g^{23}F_{24} + g^{33}F_{34}\right) = 0 \quad \text{of order } \varrho_2 \geq n - 1. \tag{41}$$

Since, due to (14), (15), and to $|g_{ik}| \neq 0$, one has also $|g_{ab}| \neq 0$, from (40), (41) it follows that F_{24} and F_{34} must present zeros of order ϱ $\left(\varrho \geq \frac{n-1}{2}\right)$. But for T^2_2 (and similarly for T^3_3) due to (38) this means:

$$T^2_2 \sim \frac{1}{V^2}\left(g^{22}F_{24}F_{24} + g^{23}F_{34}F_{34}\right) \sim (x^1)^{2\varrho - n} = (x^1)^{\alpha - 1},\ \alpha \geq 0. \tag{42}$$

Therefore also T^2_2 and T^3_3 have at most a pole of first order. E^2_2 and E^3_3 contain instead a pole of second order. Because the latter vanish, since $g^{11} \neq 0$ and since (39) holds, it must be

$$\frac{n}{2}\left(\frac{n}{2} - 1\right) = 0, \quad i.e.\ n = 2 \quad (n = 0 \text{ too}). \tag{43}$$

Static solutions of Einstein's field equations with a matter tensor for the physical fields can therefore contain only hypersurfaces S on which g vanishes to second order.

We notice now that, since the $F_{\mu\nu}$ and their first derivatives are limited by hypothesis, the null points of the components F_{k4} must have an order

$\varrho \geq 1$. The latter result follows also directly from Maxwell's field equations, if we observe that the current j_μ is finite, because it entails only covariant quantities. From

$$F_{[a4,1]} = 0,$$

since $F_{a1,4} = 0$, it turns out that the null point of F_{a4} must be of order $\varrho \geq 1$, because $F_{14,a}$ has a zero for $x^1 = 0$ and hence it is limited. Furthermore

$$g^{\mu\nu} F_{\mu 4;\nu} = j_4 = g^{11} F_{14;1} + g^{ab} F_{a4;b}$$

cannot be singular, and F_{14} must have a zero of order $\varrho \geq 1$. By inserting this result in T^l_k (38), one immediately sees (with $V^2 \sim (x^1)^n$, $n = 2$) that no first order poles appear in T^l_k, hence they must be mutually compensated in E^l_k. Therefore in general the E^l_k are finite for $x^1 = 0$.

Now, in the derivation of equations (52), (53) and (54), (55) of I, it has only been availed of the fact that $R_{\mu\nu}$ is finite. These formulae hold here too, and with the hypothesis (16) one finds again:

$$\Gamma^1_{ab} = -\frac{1}{2} g^{11} g_{ab,1} = \alpha^1_{1\ ab}(x^2, x^3) x^1 + \cdots. \tag{44}$$

Furthermore, according to I (54), (55):

$$\Gamma^1_{1a} = \alpha^1_{1\ 1a}(x^2, x^3) x^1 + \cdots, \tag{45}$$

$$\Gamma^1_{11} = \alpha^1_{1\ 11}(x^2, x^3) x^1 + \cdots. \tag{46}$$

Since $g^{11} \neq 0$, from here it follows:

$$g_{ab} = \alpha_{0\ ab}(x^2, x^3) + \alpha_{2\ ab}(x^2, x^3)(x^1)^2 + \cdots \tag{47}$$

and

$$g_{11} = \text{const.} + \alpha_{2\ 11}(x^2, x^3)(x^1)^2 + \cdots. \tag{48}$$

Therefore all the derivatives $g_{\mu\nu,1}$ vanish like x^1 for $x^1 \to 0$. Hence the introduction of the matter fields does not entail any new result with respect to the case of the vacuum dealt with in I.

3. The incompleteness of the spacetime manifold with null points of g

We show that from the structure of $g_{\mu\nu}$ found in I and in §1 it follows:

If a static metric fulfills Einstein's field equations for vacuum, then due to the null point of the determinant of $g_{\mu\nu}$ the space is necessarily incomplete, *i.e.* timelike geodesics exist with a finite branch[6]. A test particle that moves along such a geodesic, and hence is endowed with a finite rest mass, reaches the hypersurface S in a finite proper time, corresponding to an infinite system time. The test particle leaves the physical V_4. - The same result occurs also when the equations of §2 hold instead of the vacuum equations.

[6] see W. Rinow, Deutsche Mathematik, **1**, 46 (1936).

For the sake of a simplified treatment of the problem we shall deal first with the particular case of the vanishing of g in a regular, spherically symmetric vacuum metric (Einstein-Rosen spacetime[7]). According to §1 it can be cast into the form

$$\mathrm{d}s^2 = g_{11}\mathrm{d}\varrho^2 + g_{22}\left(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2\right) + \varrho^2\mathrm{d}t^2. \tag{49}$$

g_{11} and g_{22} are functions of ϱ only. They are continuous, limited and non-vanishing on the hypersurface $\varrho = 0$. One has

$$g = g_{11}(g_{22})^2\varrho^2\sin^2\vartheta, \quad |g|_{\varrho=0} = 0, \quad g_{11} < 0, \quad g_{22} < 0.$$

Since the metric does not depend on time, the equations of the geodesics have the first integral

$$\varrho^2\frac{\mathrm{d}t}{\mathrm{d}\tau} = \beta = \text{const.} \neq 0; \tag{50}$$

$\mathrm{d}\tau^2 = \mathrm{d}s^2$ for timelike geodesics. Due to the spherical symmetry, there are geodesics with $\vartheta = \text{const.}$, $\varphi = \text{const.}$; then the normalisation condition gives:

$$g_{11}\left(\frac{\mathrm{d}\varrho}{\mathrm{d}\tau}\right)^2 + \varrho^2\left(\frac{\mathrm{d}t}{\mathrm{d}\tau}\right)^2 = 1. \tag{51}$$

By keeping into account (50) and by solving one obtains:

$$\frac{\mathrm{d}\tau}{\mathrm{d}\varrho} = \varrho\sqrt{\frac{g_{11}}{\varrho^2 - \beta^2}}, \tag{52}$$

$$\frac{\mathrm{d}t}{\mathrm{d}\varrho} = \frac{\beta}{\varrho}\sqrt{\frac{g_{11}}{\varrho^2 - \beta^2}}. \tag{53}$$

Since $\beta^2 > 0$, the inequality $\beta^2 - \varrho^2 > 0$ holds in a finite neighbourhood of the hypersurface $\varrho = 0$. Therefore $\frac{\mathrm{d}\tau}{\mathrm{d}\varrho}$ has a zero and $\frac{\mathrm{d}t}{\mathrm{d}\varrho}$ has a pole of first order for $\varrho = 0$. Hence

$$\tau_1 - \tau_0 = \int_0^{\varrho_1} \varrho\mathrm{d}\varrho\sqrt{\frac{g_{11}}{\varrho^2 - \beta^2}} \tag{54}$$

is finite, while

$$t_1 - t_0 = \int_0^{\varrho_1} \frac{\beta\mathrm{d}\varrho}{\varrho}\sqrt{\frac{g_{11}}{\varrho^2 - \beta^2}} \tag{55}$$

diverges. Therefore the timelike geodesics in a spatial radial direction have a branch of finite overall length.

For a generic static field (in vacuum or in presence of scalar, spinor and Maxwell fields), on the basis of the field equations, according to I and to the equations (14), (15), (16), in the neighbourhood of S the series expansion for the $g_{\mu\nu}$ reads:

$$g_{11} = \alpha_{0\ 11} + \alpha_{2\ 11}(x^2, x^3)(x^1)^2, \ g_{44} = (x^1)^2.$$

[7]A. Einstein and N. Rosen, l.c. in footnote 2.

Therefore both $g_{11,a}$ and g_{44} have a zero of second order for $x^1 = 0$. The equations of the geodesic line still have the first integral:

$$(x^1)^2 \frac{\mathrm{d}x^4}{\mathrm{d}\tau} = \beta > 0. \tag{56}$$

For timelike geodesics it holds now the normalisation condition
(57)

$$g_{11}\left(\frac{\mathrm{d}x^1}{\mathrm{d}\tau}\right)^2 + g_{ab}u^a u^b = -\frac{\beta^2}{(x^1)^2} + 1, \ \left(u^a = \frac{\mathrm{d}x^a}{\mathrm{d}\tau}, u_b = g_{b\mu}u^\mu = g_{ba}u^a\right).$$

By solving for $\frac{\mathrm{d}x^1}{\mathrm{d}\tau}$ one finds

$$\frac{\mathrm{d}x^1}{\mathrm{d}\tau} = \sqrt{\frac{1}{g_{11}}\left(1 - \frac{\beta^2}{(x^1)^2} - g_{ab}u^a u^b\right)}. \tag{58}$$

By putting x^1 in evidence as parameter one gets:

$$\frac{\mathrm{d}\tau}{\mathrm{d}x^1} = x^1 \frac{\sqrt{-g_{11}}}{N}, \tag{59}$$

$$\frac{\mathrm{d}t}{\mathrm{d}x^1} = \frac{\beta}{x^1}\frac{\sqrt{-g_{11}}}{N}, \tag{60}$$

with $N = \left(\beta^2 - (x^1)^2 + (x^1)^2 g_{ab}u^a u^b\right)^{1/2}$. Due to the normalisation and since g_{11}, $g_{ab} < 0$ one has

$$0 \leq |N| \leq |\beta|. \tag{61}$$

Equations (59), (60) have the same consequence as (52), (53), if one can show that there are geodesics with $0 < |N|$. To this end it is sufficient that $g_{ab}u^a u^b$ remain finite.

The equations for u^a read:

$$\frac{\mathrm{d}u_a}{\mathrm{d}\tau} - \frac{1}{2}g_{11,a}\left(\frac{\mathrm{d}x^1}{\mathrm{d}\tau}\right)^2 - \frac{1}{2}g_{bc,a}u^b u^c = 0, \tag{62}$$

$$\frac{\mathrm{d}u_a}{\mathrm{d}\tau} - \frac{1}{2}\frac{g_{11,a}}{g_{11}}\left[1 - \frac{\beta^2}{(x^1)^2} - g_{bc}u^b u^c\right] - \frac{1}{2}g_{bc,a}u^b u^c = 0. \tag{63}$$

This is a regular system of differential equations for u_a, since $g_{11,a}$ has a zero of second order for $x^1 = 0$. Because there exist geodesics that satisfy the initial conditions $u_a = 0$ arbitrarily close to the hypersurface $x^1 = 0$, there exist geodesics with finite u^a and hence with finite $g_{ab}u^a u^b$. Due to (59), (60)

$$\tau_1 - \tau_0 = \int_0^a \mathrm{d}x^1 \frac{\mathrm{d}\tau}{\mathrm{d}x^1} \tag{64}$$

is then finite, and

$$t_1 - t_0 = \int_0^a \mathrm{d}x^1 \frac{\mathrm{d}t}{\mathrm{d}x^1} \tag{65}$$

is divergent. Therefore the existence of timelike geodesics with a finite branch is warranted.

According to what has been told in §1, $x^1 = \varrho$ must be interpreted as a radial coordinate; the $x^a(a = 2, 3)$ can be interpreted as angular coordinates.

Let us consider now a test particle moving along a geodesic that enters the region around $\varrho = 0$. For small ϱ, u^a is limited; therefore $|N| > 0$, and the worldline of the test particle is a geodesic with a "finite branch". The particle asymptotically approaches the worldline $\varrho = 0$, without reaching it in a finite system time, while it abandons V_4 in a finite proper time. For an observer at infinity, for whom the system time is coincident with his proper time, the test particle will then be absorbed by the particle of finite mass, represented by $g = 0$.

If N has a zero for $\varrho = 0$, u^a must diverge like $1/x^1$. The particle approaches the worldline $\varrho = 0$ asymptotically, and in the three-dimensional space it describes a spiral that asymptotically approaches $\varrho = 0$. The angular velocity then obviously diverges also with respect to the system time. Since the particle does not go far again from $\varrho = 0$, for an observer staying at infinity, it is absorbed too. But also the proper time of the particle can diverge, hence in this case the branch of geodesic does not have a finite overall length.

If N has a zero for $x^1 = \varepsilon$, where $\varepsilon > 0$ is a small number, there it is $\mathrm{d}x^1/\mathrm{d}\tau = 0$ due to (58). When $\mathrm{d}^2x^1/\mathrm{d}\tau^2 \neq 0$ the particle may invert its path in this point. But if u^a is very large, it describes in a finite proper and system time a spiral around $\varrho = 0$, before getting again far from the neighbourhood of $\varrho = 0$.

One understands that the vanishing of g and the consequent incompleteness of V_4 can be recognised in principle through diffusion experiments: when the region with higher gravitational field strength is entered, *i.e.* the neighbourhood of $\varrho = 0$ is reached, one observes absorption and irregular diffusion of the test particle, that do not happen in a complete spacetime manifold.

Therefore in principle, through diffusion experiments, it is possible to distinguish between a complete form of a spherically symmetric gravitational field and the Einstein-Rosen space, although all the spherically symmetric Einstein spaces are locally isometric[8].

4. Conclusions

We have seen in §2 that also for static gravitational fields, that are coupled with matter, only zeros of second order of g on S are possible. These

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[8]On the complete Schwarzschild field see M. Kruskal, Physic. Rev. **119**, 1743 (1960). Through diffusion experiments it should be possible in principle to tell apart also the Einstein-Rosen metric from the usual Schwarzschild metric, because the Einstein-Rosen space and the Schwarzschild space are endowed with a different global structure.

spaces describe always pointlike particles without an internal structure. S degenerates to a worldline.

It is consequent to consider now spaces exhibiting time-dependent gravitational fields. In fact only with a dependence on time a change of signature can really be effected in the region with $g \geq 0$. For gravitational fields not dependent on time Einstein's equations are everywhere elliptic (in three dimensions). In the time-dependent case, they are hyperbolic in three dimensions in the external region with $g < 0$, $s = -2$. They are parabolic for $g = 0$, $s = -3$. For $g > 0$, $s = -4$ they are elliptic equations in four dimensions.

A difficulty of principle in dealing with time-dependent gravitational fields with regions where $g \geq 0$ $(s = -4, -3)$ is the invariant definition of the hypersurfaces S on which g vanishes. - Our definition of S in I and in §1 presupposes the existence of a timelike Killing vector for $g < 0$. - But if S is not introduced in an invariant way, there is no problem, through a transformation aptly nonregular for $\overline{x}^1 = 0$ (with $\overline{x}^\nu = \overline{x}^\nu(x^\mu)$ possibly complex for $\overline{x}^1 < 0$) in bringing always a regular, locally static field $g_{\mu\nu}$ with $g \neq 0$ in the form

$$\overline{g}_{00} = (\overline{x}^1)^n, \ \overline{g}_{i0} = 0, \ n > 1, \tag{66}$$

$$\overline{g}_{ik} = \overline{g}_{ik}(\overline{x}^\nu) \text{ with } |\overline{g}_{ik}| \neq 0, \tag{67}$$

as one easily gathers from the transformation formulae for the $g^{\mu\nu}$. - One interprets the $\overline{x}^\nu$ as the regular coordinates that define the metric topology of V_4, and $\overline{g}_{\mu\nu}$ is a regular gravitational field with null points of $\overline{g}$.

For fields periodic in time an invariant, also physically privileged definition of S appears possible, because in this case a time direction is privileged too. For these fields one can obtain

$$g_{\mu\nu}(x^i, x^4) = g_{\mu\nu}(x^i, x^4 + mT)$$

with m = integer number, T = period. - But due to the results of Papapetrou[9] in the weak field region no field periodic in time is compatible with Einstein's vacuum equations, and the periodicity in time must be limited to a finite spatial region. According to Papapetrou and Treder[10] the latter must be separated from the external field, not dependent on time, by a null surface Σ, "globally isolated". Within Σ the gravitational field is strong and S must stay within Σ.

The results of §3 appear to be of fundamental meaning. The incompleteness of the static Einstein spaces with the hypersurface S degenerated in a worldline $g = 0$ allows for the existence of everywhere regular static solutions, that asymptotically coincide with the space of Minkowski, and that

[9]A. Papapetrou, Ann. Physik (6), **20**, 399 (1957).

[10]A. Papapetrou and H. Treder, Ann. Physik (7), **3**, 360 (1959).

describe particles endowed with rest mass[11]. The known proof of the nonexistence of such solutions by Einstein and Pauli, R. Serini and A. Lichnerowicz is in fact directly based on the hypothesis of completeness for the spacetime manifold[12],[13].

If we have a regular, static Einstein space V_4 with boundary conditions (when one maintains all the other usual topologic hypotheses), then it is incomplete and, in keeping with Einstein's proposal, the particles are represented through a worldline $g = 0$. The completion of V_4, *i.e.* the introduction of a new space V_4^*, locally isometric to V_4 - except along the worldline $g = 0$ - that is everywhere endowed with the indefinite signature $s = -2$, necessarily brings to the fact that V_4^* contains a worldline on which Einstein's equations are not satisfied, because it appears on it a singular (delta-like) matter distribution.

From §§1 and 2 it follows that V_4 and V_4^*, despite their local isometry, due to their different global structure have distinct physical properties, that can be recognised through diffusion experiments.

In general, within Einstein's particle problem, the following program for a field-theoretical model of discrete matter (particles) appear possible.

A complete V_4^* with discrete matter distribution is assumed. In it finite spatial regions with $R_{\mu\nu} \neq 0$ are present; elsewhere one has $R_{\mu\nu}$=0. The $g_{\mu\nu}$ have the index of inertia $(-1, -1, -1, +1)$ and asymptotically approach the Minkowskian values. - One now defines an incomplete V_4 in such a way that V_4 is an Einstein space with regular $g_{\mu\nu}$, locally isometric to V_4^* in the region where V_4^* fulfills the vacuum equations. V_4, however, contains also regions where $g \geq 0$. The latter correspond to the regions[14] in which $R_{\mu\nu} \neq 0$.

Due to the supposed incompleteness of Einstein's V_4 and to the completeness of the original V_4^*, the two spacetimes can be distinguished physically despite their local isometry. A further discussion will follow in a subsequent Communication.

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[11]see H. Treder, l.c. in footnote 1.

[12]A. Einstein and W. Pauli, Ann. Math. **44**, 131 (1943); A. Lichnerowicz, Theories relativistes de la Gravitation, Paris, 1955.

[13]Complete static Einstein spaces are not compatible with the requirements that $g_{\mu\nu}$ be regular and asymptotically Minkowskian also when in the region of strong fields (like for an incomplete V_4) the character of the Killing vector changes; see A. Papapetrou and H. Treder, Comptes Rend. Acad. Sci. Paris **254**, 4254 (1962). - A further discussion appears in the Mathematischen Nachrichten.

[14]An analogous situation occurs when, like in §2, the gravitational field is coupled with matter fields, but no a posteriori (phenomenologic) inhomogeneities are allowed for, as required by the program of a pure field theory. - In fact, according to Einstein and Pauli (see l.c. in footnote 2) and to Y. Thiry, Journ. math. pures et appl. (9) **30**, 275 (1951), for time-dependent Einstein-Maxwell fields in a complete V_4^* theorems hold that are analogous to the ones that hold in Einstein's vacuum field.